Halfspace Depth and Floating Body

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Halfspace depth and floating body, STATISTICS SURVEYS *2019, Vol. 13, 52-118.*

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Notions and Definitions

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- H denotes a hyperplane in $\mathbb{R}^n$ and H^- and H^+ its halfspaces.
- The volume of a convex body K is denoted by $\text{vol}_n(K)$. It is the Lebesgue measure of K.

Floating body

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Let K_δ be the intersection of all halfspaces whose defining hyperplanes cut off a set of volume δ from K. i.e.

$$K_\delta = \bigcap_{\mathrm{vol}_n(K \cap H^-) = \delta} H^+$$

K_δ is called the convex floating body of K for index δ.

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Theorem (Meyer-Reisner)

Every centrally symmetric convex body has floating bodies for all δ.

Applications of floating body

Affine surface area

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Affine surface area

$$\lim_{\delta \to 0} \frac{\mathrm{vol}_n(K) - \mathrm{vol}_n(K_\delta)}{\delta^{\frac{2}{n+1}}} = c_n \int_{\partial K} \kappa^{\frac{1}{n+1}} d\mu_{\partial K}$$

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Blaschke-Santaló inequality

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For multivariate data/ random variables there is no natural ordering.

Halfspace depth

Definition (Tukey, Donoho, Gasko)

Let $\mathbb{P}$ be a probability measure on $\mathbb{R}^n$. The halfspace depth or Tukey depth hD of $\mathbb{P}$ at x is

$$hD(x,\mathbb{P}) = \inf\{\mathbb{P}(H^-)|H \text{ is a hyperplane with } x \in H\}.$$

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Zuo, Serfling, Dyckerhoff, Mosler, Rousseeuw, Ruts

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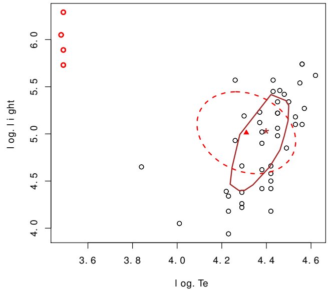
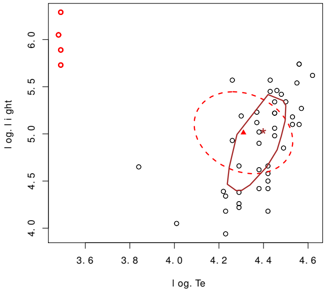
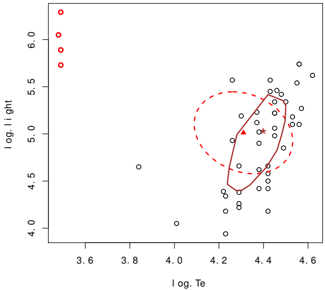
Let $\mathbb{P}$ be the uniform measure on the vertices of a simplex in $\mathbb{R}^n$.
Then all elements of the simplex are halfspace medians of $\mathbb{P}$.



The Hertzsprung-Russell diagram of the Star Cluster CYG OB1.



The Hertzsprung-Russell diagram of the Star Cluster CYG OB1. The four giant stars (red points) attract both the sample mean (red triangle), and the sample covariance (represented by the red ellipse).



The Hertzsprung-Russell diagram of the Star Cluster CYG OB1. The four giant stars (red points) attract both the sample mean (red triangle), and the sample covariance (represented by the red ellipse). The halfspace depth-based median (brown star), and the depth-based central region containing 25 % of the observations, provide a more appropriate representation of the location and the variability of the main data cloud.

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$$\mathbb{P}(H) = 0.$$

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$$\mathbb{P}(H_1^-) > 0 \qquad \mathbb{P}(H_2^-) > 0$$

and

$$\mathbb{P}(H_1^-) + \mathbb{P}(H_2^-) = 1.$$

Lemma

If $\mathbb{P}$ has contiguous support then the halfspace median is unique.

Example

Let $0 < \alpha \leq 2$ and $\phi : \mathbb{R} \to \mathbb{R}$ a continuous function. Let X be a random vector such that for all $t \in \mathbb{R}^n$

$$\mathbb{E}e^{i\langle t,X\rangle} = \phi(\|t\|_\alpha).$$

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Then we say that X is α-symmetric. $\mathbb{P}_X$ is the joint distribution. In this case we can compute the central regions/ floating bodies of $\mathbb{P}_X$.

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$$\begin{aligned} hD(x,\mathbb{P}_X) &= \inf\{\mathbb{P}_X(H^-)|H \text{ is a hyperplane with } x\in H\} \\ &= \inf_{t\neq 0}\mathbb{P}(\langle t,X\rangle\leq\langle t,x\rangle)=\inf_{t\neq 0}\mathbb{P}(\|t\|_\alpha X_1\leq\langle t,x\rangle) \end{aligned}$$

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We have

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$$\alpha^* = \begin{cases} \frac{\alpha}{\alpha-1} & \alpha>1 \\ \infty & 0<\alpha\leq 1 \end{cases}$$

Therefore

$$hD(x,\mathbb{P}_X) = F_{X_1}(-\|x\|_{\alpha*})$$

where F_{x_1} is the distribution function of X_1.

Definition

A nonempty convex set $\mathbb{P}_{[\delta]}$ is the floating body of $\mathbb{P}$ for index δ if for each supporting hyperplane H of $\mathbb{P}_{[\delta]}$ with $H^+ \supseteq \mathbb{P}_{[\delta]}$ we have $\mathbb{P}(H^-) = \delta$.

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Definition

The convex floating body of the probability measure $\mathbb{P}$ on $\mathbb{R}^n$ with index δ is the intersection of all closed halfspaces whose defining hyperplanes cut off a set of probability content at most δ from $\mathbb{P}$, i.e.

$$\mathbb{P}_\delta^{FB} = \bigcap_{\mathbb{P}(H^-)\leq\delta} H^+.$$

Proposition

Assume that $\mathbb{P}_{[\delta]}$ exists. Then

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Example

Let K be a convex body in $\mathbb{R}^n$ and $\mathbb{P}_K$ the uniform measure on K. Then

$$K_\delta = (\mathbb{P}_K)_\delta.$$

Question

Do the central regions/ floating bodies of a measure determine the measure?

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Partial results: Struyf, Rousseeuw, Koshevoy, Hassairi, Regaieg, Kong, Zuo

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Consider the α-symmetric random vector $X = (X_1, \ldots, X_n)$ with $\alpha = 1$ and $\phi(\theta) = \sqrt{\theta}$

$$\psi_X(t) = \exp\left(-\|t\|_1^{\frac{1}{2}}\right) = \exp\left(-\sqrt{\sum_{j=1}^{n} |t_j|}\right)$$

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and $Y = (Y_1, \ldots, Y_n)$ with $\alpha = \frac{1}{2}$ and $\phi(\theta) = \sqrt{\theta}$

$$\psi_Y(t) = \exp\left(-\|t\|_{\frac{1}{2}}^{\frac{1}{2}}\right) = \exp\left(-\sum_{j=1}^{n} \sqrt{|t_j|}\right)$$

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Clearly, $\mathbb{P}_X$ and $\mathbb{P}_Y$ are different.

For every $x \in \mathbb{R}^n$ the halfspace depth are the same

$$hD(x, \mathbb{P}_X) = F_{X_1}(-\|x\|_\infty) \qquad\qquad hD(x, \mathbb{P}_Y) = F_{Y_1}(-\|x\|_\infty)$$

where F_{X_1} is the distribution function of X_1 and F_{Y_1} is the distribution function of Y_1.

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where F_{X_1} is the distribution function of X_1 and F_{Y_1} is the distribution function of Y_1. But $F_{X_1} = F_{Y_1}$

$$\psi_{X_1}(t_1) = \exp(-\sqrt{|t_1|}) = \psi_{Y_1}(t_1)$$

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(i) For each $\delta \in (0, \frac{1}{2})$ the floating body $\mathbb{P}_{[\delta]}$ of $\mathbb{P}$ exists.

(ii) For all hyperplanes H we have $\mathbb{P}(H) = 0$ and

$$\mathbb{P}(H^-) = \begin{cases} \sup_{x \in H} hD(x, \mathbb{P}) & x_{\mathbb{P}} \notin H^- \\ 1 - \sup_{x \in H} hD(x, \mathbb{P}) & x_{\mathbb{P}} \in H^- \end{cases}$$

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Consequently, if (i) is true then $\mathbb{P}$ is characterized by its halfspace depth.

Theorem

Suppose that for all hyperplanes H we have $\mathbb{P}(H) = 0$ and that $\mathbb{P}$ has contiguous support. Then the map $\delta \to \mathbb{P}_\delta$ is continuous w.r.t. the Hausdorff distance.

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Question

For which $\mathbb{P}$ do the central regions/ floating bodies have a C^1- or C^2-boundary?