Functions with isotropic sections

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The problem

Problem

Assume that for a measurable subset U of S^{n-1} and for an even bounded measurable function $g : S^{n-1} \to \mathbb{R}$, the restriction $g|_{S^{n-1}\cap u^\perp}$ onto $S^{n-1} \cap u^\perp$ is isotropic, for almost all $u \in U$. What can be said about g?

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Results

Theorem

Let U be an open subset of $\mathbb{S}^{n-1}$, that does not contain $U^{\perp}$. There exists a continuous function $g : \mathbb{S}^{n-1} \to \mathbb{R}$, such that for any $u \in U$, $g|_{\mathbb{S}^{n-1} \cap u^{\perp}}$ is isotropic, but g is not constant on $U^{\perp}$.

Funk and cosine transform of a function $\zeta : \mathbb{S}^{n-1} \to \mathbb{R}$:

$$\mathcal{R}(\zeta)(u) = \int_{\mathbb{S}^{n-1} \cap u^{\perp}} \zeta(x)dx, \qquad u \in \mathbb{S}^{n-1},$$

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Let $n \geq 3$, U be an open subset of $\mathbb{S}^{n-1}$ and $g : U \to \mathbb{R}$ be an even, bounded, measurable function. If for almost every $u \in U$, $g|_{\mathbb{S}^{n-1} \cap u^{\perp}}$ is isotropic, then $C(g)|_U = c + \langle a, \cdot \rangle$ and $\mathcal{R}(g)|_U = c'$, almost everywhere in U, for some fixed constants $c, c' \in \mathbb{R}$ and for some fixed vector $a \in \mathbb{R}^n$.

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- In general, $\frac{dS_i(Z(g),\cdot)}{dx} = c_{n,i} \int_{\mathbb{S}^{n-1}\cap u^\perp} \cdots \int_{\mathbb{S}^{n-1}\cap u^\perp} \det(x_1, \dots, x_{n-1})^2 g(x_1) \dots g(x_i) dx_1 \dots dx_{n-}$

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- Radii of curvature of a smooth convex body K at $u \in \mathbb{S}^{n-1}$: Eigenvalues of the matrix $(h_{ij} + h\delta_{ij})_{i,j=1}^{n-1}$ at u
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- Recall: $\frac{dS_1(Z(g),\cdot)}{dx} = (r_1 + \cdots + r_{n-1})/(n-1)$ and $\frac{dS_{n-1}(Z(g),\cdot)}{dx} = r_1 \dots r_{n-1}$

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The counterexample

- Take G to be a C^{∞} function on the sphere, which is zero at U (but not identically equal to zero).
- There exists continuous $w : \mathbb{S}^{n-1} \to \mathbb{R}$, such that $G(u) = C(w)(u) = \int_{\mathbb{S}^{n-1}} |\langle x, u \rangle| w(x) dx$.
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Regularity issues

Theorem

Let K be a convex body in $\mathbb{R}^n$, $n \geq 3$, U be an open connected subset of $\mathbb{S}^{n-1}$ and assume that the measure $S_1(K,\cdot)|_{\mathcal{B}(U)}$ is absolutely continuous. If for almost every direction $u \in U$ it holds

$$r_K^1(u) = \cdots = r_K^{n-1}(u),$$

then $\tau(K,U)$ is contained in a Euclidean sphere.

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