A Functional version of the Busemann-Petty centroid inequality.

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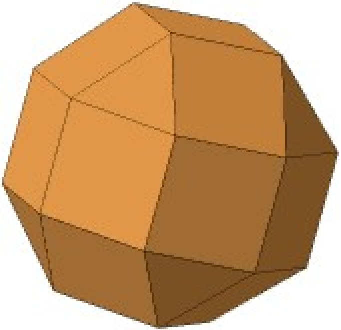
Convex bodies

The main objects of study are convex bodies. A convex body is a subset $K \subseteq \mathbb{R}^n$ which is convex, compact and has non-empty interior.



Associated functionals

For $K \subset \mathbb{R}^n$ as before, its support function, its gauge (or Minkowski functional) and its radial function are defined respectively by

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Clearly, $\|x\|_K = \frac{1}{r_K(x)}$

Associated bodies

Polar body

For $K \subset \mathbb{R}^n$ we define its polar body, denoted by K°, by

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norm	body	polar body	dual norm
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Lutwak and Zhang introduced for a body K its L_p-Centroid body denoted by $\Gamma_p K$. This body is defined by

$$h^p_{\Gamma_p K}(x) := \frac{1}{c_{n,p}\,\mathrm{vol}(K)} \int_K |\langle x, y\rangle|^p dy \quad \text{for } x \in \mathbb{R}^n,$$

where

$$c_{n,p} = \frac{\omega_{n+p}}{\omega_2 \omega_n \omega_{p-1}}, \quad \omega_k = \mathrm{vol}(B_2^k),$$

connected to this we also have the L_p-Moment body of K denoted by $M_p K$ and defined via

$$h_{M_p K}(x)^p = \int_K |\langle x, y\rangle|^p dy,$$

Related inequalities

L_p Busemann-Petty centroid inequality

$$\mathrm{vol}(\Gamma_p K) \geq \mathrm{vol}(K) \quad \text{(Lutwak, Yang and Zhang)}\ .$$

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In terms of the Moment body $M_p K$ we have

$$\mathrm{vol}(M_p K) \geq c_{n,p}^{n/p} \, \mathrm{vol}(K)^{\frac{n+p}{p}}$$

Euclidean Inequalities

(Aubin and Talenti)

$$\|f\|_{\frac{np}{n-p}} \leq \mathcal{S}_{n,p} \left(\int_{\mathbb{R}^n} |\nabla f|^p dx \right)^{1/p}$$

$$f(x) = \left(a + b|x - x_0|^{\frac{p}{p-1}} \right)^{1-\frac{p}{n}}$$

(Del Pino-Dolbeault)

$$\mathrm{Ent}(|f|^p) = \int |f|^p \log |f|^p dx \leq \frac{n}{p} \log \left(\mathcal{L}_p \int |\nabla f|^p dx \right),$$

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Inequalities with an abstract norm

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Lutwak-Yang-Zhang (L_p Minkowski problem + L_p Petty Projection Ineq.)

$$\|f\|_{\frac{np}{n-p}} \leq \mathcal{S}_{n,p}\left(c_{n,p}\int_{S^{n-1}}\|\nabla_\xi f\|_p^{-n}d\xi\right)^{-1/n}$$

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Haberl, Schuster and Xiao and independently Zhai

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Applications: some inequalities

Sobolev

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Log - Sobolev

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Gagliardo-Nirenberg

$$\|f\|_r \leq \mathcal{G}_{n,p,m,r} \|\nabla f\|_p^\theta \|f\|_m^{1-\theta}$$

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$$\mathrm{Ent}(e^{\beta f}) \leq n \log \left(\frac{\beta k_n}{e} \|\nabla f\|_\infty \right)$$

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$$\left(\int_{\partial\mathbb{R}^n_+} |f(0,x)|^{\frac{p(n-1)}{n-p}} dx\right)^{\frac{n-p}{p(n-1)}} \leq \mathcal{K}_{n,p} \left(\int_{\mathbb{R}^n_+} |\nabla f(t,x)|^p dxdt\right)^{\frac{1}{p}}$$

Weighted Sobolev

$$\left(\int_{\mathbb{R}^n_+} |f(t,x)|^{\frac{np}{n-p}} t^a dxdt\right)^{\frac{n-p}{np)}} \leq \mathcal{K}_{n,p,a} \left(\int_{\mathbb{R}^n_+} |\nabla f(t,x)|^p t^a dxdt\right)^{\frac{1}{p}}$$

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$$\|f\|_{L^{\alpha p}(\mathbb{R}^n_+,\omega)} \leq \left(\int_{\mathbb{R}^n_+} |\nabla f(t,x)| t^a dy\right)^{\frac{\theta}{p}} \|f\|^{1-\theta}_{L^{\alpha(p-1)+1}(\mathbb{R}^n_+,\omega)}$$

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Busemann-Petty centroid

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Mixed volume

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$$V_r(K, L) = \frac{r}{n} \lim_{\varepsilon \to 0} \frac{\mathrm{vol}(K +_r \varepsilon \cdot_r L) - \mathrm{vol}(K)}{\varepsilon},$$

where $K +_r \varepsilon \cdot_r L$ is the convex body defined by:

$$h_{K +_r \varepsilon \cdot_r L}(x)^r = h_K(x)^r + \varepsilon h_L(x)^r, \quad \forall x \in \mathbb{R}^n.$$

It was shown by E. Lutwak that there exists a unique finite positive Borel measure $S_r(K,.)$ on $\mathbb{S}^{n-1}$ such that

$$V_r(K,L) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_L(u)^r dS_r(K,u), \tag{1}$$

for each convex body L. In the same work he provided the following special case of L_p Minkwoski inequality for mixed volumes.

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for each convex body L. In the same work he provided the following special case of L_p Minkwoski inequality for mixed volumes.
If $1 \le r < \infty$ and K, L are convex bodies in $\mathbb{R}^n$ containing the origin as interior point, then

$$V_r(L,K) \ge \mathrm{vol}(L)^{\frac{n-r}{n}}\, \mathrm{vol}(K)^{\frac{r}{n}}. \tag{2}$$

Geometric Inequalities

Combining

$$V_r(L,K) \geq \mathrm{vol}(L)^{\frac{n-r}{n}} \, \mathrm{vol}(K)^{\frac{r}{n}}.$$

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we can write

$$V_r(L, M_pK) \geq c_{n,p}^{r/p} \mathrm{vol}(L)^{\frac{n-r}{n}} \mathrm{vol}(K)^{\frac{(n+p)r}{np}}. \qquad (3)$$

which for $L = M_pK$, using the well-known fact that $V_r(L,L) = \mathrm{vol}(L)$, reduces to the L_p-Busemann-Petty centroid inequality mentioned above.

Functional inequalities

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We want to define something reasonable for $V_r(f, M_p g)$. Let us look back at this integral representation

$$\begin{aligned} V_r(L, M_p K) &= \frac{1}{n} \int_{S^{n-1}} h_{M_p K}(u)^r dS_r(L, u) \\ &= \frac{1}{n} \int_{S^{n-1}} \left(\int_K |\langle u, z \rangle|^p dz \right)^{r/p} dS_r(L, u) \\ &= \frac{1}{n} \int_{S^{n-1}} \left(\int_{\mathbb{R}^n} 1_K(z) |\langle u, z \rangle|^p dz \right)^{r/p} dS_r(L, u). \end{aligned}$$

The surface area measure of a function

The L^r surface area measure of a function $f : \mathbb{R}^n \to \mathbb{R}$ with L^r weak derivative is given by the lemma:

Lemma (LYZ)

Given $1 \leq r < \infty$ and a function $f : \mathbb{R}^n \to \mathbb{R}$ with L^r weak derivative, there exists a unique finite Borel measure $S_r(f, .)$ on $\mathbb{S}^{n-1}$ such that

$$\int_{\mathbb{R}^n} \phi(-\nabla f(x))^r dx = \int_{\mathbb{S}^{n-1}} \phi(u)^r dS_r(f, u), \tag{4}$$

for every nonnegative continuous function $\phi : \mathbb{R}^n \to \mathbb{R}$ homogeneous of degree 1. If f is not equal to a constant function almost everywhere, then the support of $S_r(f, .)$ cannot be contained in any $n - 1$ dimensional linear subspace.

1 - Erwin Lutwak, Deane Yang, and Gaoyong Zhang. Optimal Sobolev norms and the lp Minkowski problem. International Mathematics Research Notices, 2006, 2006.

Functional mixed volume

In view of the previous identity we have that for any f and L such that $S_r(f,.) = S_r(L,.)$, we have

$$V_r(L,K) = \frac{1}{n}\int_{\mathbb{R}^n} h_K(-\nabla f(x))^r dx.$$

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Definition

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Definition: If g is a nonnegative function with compact support, we define the convex body $M_p g$ by

$$h(M_p g, \xi)^p = \int_{\mathbb{R}^n} g(x)|\langle x, \xi\rangle|^p dx. \tag{6}$$

Functional mixed volume

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Definition: If f is a C^1 function and g is nonnegative, we define the r-functional mixed volume of f and $M_p g$ by:

$$V_r(f, M_p g) = \frac{1}{n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} g(y) |\langle \nabla f(x), y \rangle|^p dy \right)^{r/p} dx. \tag{7}$$

Main Results

Theorem (Haddad, J., Silva)

Let f be a C^1 function and g a continuous non-negative function, both with compact support in $\mathbb{R}^n$, then for $1 \leq r < n$, $q = \frac{nr}{n-r}$ and $\lambda \in \left(\frac{n}{n+p}, 1\right) \cup (1, \infty)$,

$$V_r(f, M_p g) \geq c_3^{\frac{r}{p}} ||g||_1^{\frac{[(n+p)(\lambda-1)+p]r}{np(\lambda-1)}} ||g||_\lambda^{-\frac{\lambda r}{n(\lambda-1)}} ||f||_q^r. \tag{8}$$

where $c_3 = c_1^p c_2^{-\frac{(n+p)(\lambda-1)+p}{n(\lambda-1)}}$.

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Main Results

Our main result will be a consequence of the following two theorems

Theorem

If f is a C^1 function with compact support in $\mathbb{R}^n$ and K symmetric convex body, then for $1 < r < n$ and $q = \frac{nr}{n-r}$

$$V_r(f, K) \geq c_1^r ||f||_q^r \operatorname{vol}(K)^{\frac{r}{n}}, \tag{9}$$

Main Results

And

Theorem

If g is a non-negative function with compact support in $\mathbb{R}^n$, then, for each $\lambda \in \left(\frac{n}{n+p}, 1\right) \cup (1, \infty)$, we have that

$$\mathrm{vol}(M_p g)^{\frac{p}{n}} \geq c_2^{-\frac{(n+p)(\lambda-1)+p}{n(\lambda-1)}} ||g||_1^{\frac{(n+p)(\lambda-1)+p}{n(\lambda-1)}} ||g||_\lambda^{-\frac{\lambda p}{n(\lambda-1)}},$$

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For

$$G_{p,\lambda}(t) = \begin{cases} (1 + \|x\|_K^p)^{\frac{1}{\lambda-1}} & \text{if } \lambda < 1 \\ (1 - \|x\|_K^p)_+^{\frac{1}{\lambda-1}} & \text{if } \lambda > 1, \end{cases} \tag{10}$$

we recover

$$\mathrm{vol}(M_p K) \geq c_{n,p}^{n/p} \, \mathrm{vol}(K)^{\frac{n+p}{p}}$$

Ideas in the proof

For f be a C^1 function with compact support in $\mathbb{R}^n$ and $t > 0$, consider the level sets of f in $\mathbb{R}^n$:

$$N_{f,t} = \{x \in \mathbb{R}^n : |f(x)| \geq t\}$$

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We show

$$\int_0^\infty \mathrm{vol}(N_{f,t})^{\frac{n+p}{n}} dt \geq c_\lambda ||f||_1^{\frac{(n+p)(\lambda-1)+p}{n(\lambda-1)}} ||f||_\lambda^{-\frac{\lambda p}{n(\lambda-1)}},$$

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$$\begin{aligned} V_r(f, K) &= \int_0^\infty V_r(f, t, K) dt \\ &= \int_0^\infty V_r(K_t, K) dt \\ &\geq \int_0^\infty \operatorname{vol}(K_t)^{\frac{n-r}{n}} \operatorname{vol}(K)^{\frac{r}{n}} dt \\ &= \int_0^\infty \operatorname{vol}(K_t)^{\frac{n-r}{n}} dt \operatorname{vol}(K)^{\frac{r}{n}} \\ &\geq c_2^r ||f||_q^r \operatorname{vol}(K)^{\frac{r}{n}} \end{aligned}$$

Ideas in the proof

$$\mathrm{vol}(M_p g)^{\frac{p}{n}} \geq c_{n,p} a_{n,p,\lambda} ||g||_1^{\frac{(n+p)(\lambda-1)+p}{(\lambda-1)n}} ||g||_\lambda^{-\frac{\lambda p}{(\lambda-1)n}},$$

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$$\begin{aligned} \mathrm{vol}(M_p g) &= V_p(M_p g, M_p g) \\ &= \int_0^\infty V_p(M_p g, M_p N_{g,t}) dt \\ &\geq \mathrm{vol}(M_p g)^{\frac{n-p}{n}} \int_0^\infty \mathrm{vol}(M_p N_{g,t})^{p/n} dt \end{aligned}$$

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$$\mathrm{vol}(M_p g)^{\frac{p}{n}} \geq \int_0^\infty \mathrm{vol}(M_p N_{g,t})^{\frac{p}{n}} dt$$

(B-P for domains) $$\geq c_{n,p} \int_0^\infty \mathrm{vol}(N_{g,t})^{\frac{n+p}{n}} dt$$

(Technical Lemma) $$\geq c_{n,p} a_{n,p,\lambda} ||g||_1^{\frac{(n+p)(\lambda-1)+p}{(\lambda-1)n}} ||g||_\lambda^{-\frac{\lambda p}{(\lambda-1)n}},$$

Thank you for your attention!