On some theorems on the restriction of operator to coordinate subspace

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Notations:

Consider $N \times n$ matrix A as the operator from $\mathbb{R}^n$ to $\mathbb{R}^N$,

$$\|A\|_{(p,q)} = \sup_{\|x\|_{l_p^n} \leqslant 1} \|Ax\|_{l_q^N}, \ 1 \leqslant p, q \leqslant \infty$$

$$\langle N \rangle = \{1, 2, \dots, N\}$$

$$v_i, \quad i \in \langle N \rangle \quad - \quad \text{rows of } \ A$$

$$w_j, \quad j \in \langle n \rangle \quad - \quad \text{columns of } \ A$$

If $\Omega \subset \langle N \rangle$, then $A(\Omega)$ – submatrix generated by v_i, $i \in \Omega$

$(\cdot , \cdot)$ – scalar product in $\mathbb{R}^n$

$$\|x\|_p \equiv \|x\|_{l_p^n} \quad \text{if} \quad x \in \mathbb{R}^n$$

If $\Phi = \{\varphi_i(x)\}_{i=1}^{n}$ – system of functions on X, then

$$S_{\Phi}^{*}(\{a_i\}_{i=1}^{n}) = f(x) = \sup_{1 \leqslant s = s(x) \leqslant n} \left| \sum_{i=1}^{s} a_i \varphi_i(x) \right|$$

Kolmogorov's rearrangement problem (1925 ? – 1930 ?):

$$\Phi = \{\varphi_i(x)\}_{i=1}^{\infty} \text{ – O.N.S.}$$

Does it exist $\sigma \in S(\infty)$ such that

$$\sum a_i \varphi_{\sigma(i)}(x)$$

converges almost everywhere if

$$\sum a_i^2 < \infty \quad ?$$

Equivalent finite-dimensional version of this problem:

$\Phi = \{\varphi_i(x)\}_{i=1}^{n}$ – O.N.S.

Does it exist $\sigma \in S(n)$ such that for $\Phi_\sigma = \{\varphi_{\sigma(i)}\}_{i=1}^{n}$ the operator $S^*_{\Phi_\sigma}$ has bounded weak $2 \to 2$ norm?

By duality

$$\|A\|_{(2,1)} = \sup_{\varepsilon_i = \pm 1} \Big\| \sum_{i=1}^{N} \varepsilon_i v_i \Big\|_2$$

Marcus, Spielman, Srivastava (2015):

THEOREM 1. *If* $\{w_j\}_{j=1}^{n}$ *– orthonormal system in* $\mathbb{R}^N$,
$\|v_i\| \leqslant \varepsilon$, $0 < \varepsilon < 1$, $i = 1, 2, \ldots, N$. *Then*

$$\langle N \rangle = \Omega_1 \cup \Omega_2, \qquad \Omega_1 \cap \Omega_2 = \varnothing \qquad (1)$$

$$\|A(\Omega_k)\|_{(2,2)} \leqslant \frac{1}{\sqrt{2}} + C\varepsilon, \quad k = 1, 2$$

(C – absolute constant)

MAIN COROLLARY ("MSS - points")

Let $\{\varphi_i(x)\}_{i=1}^n$ – ONS $\subset L^2(X,\mu)$,

$$\frac{n}{2} \leqslant \sum \varphi_i^2(x) \leqslant 2n \qquad \forall x \in X,$$

Then $\exists\, \Omega = \{x_1, \dots, x_s\} \subset X$, $s \leqslant \mathrm{C}_1 n$,

such that for any $P = \sum a_i \varphi_i(x)$:

$$\mathrm{C}_3 \sqrt{\frac{1}{s}\sum_{\nu=1}^{s} |P(x_\nu)|^2} \leqslant \|P\|_{L^2} \leqslant \mathrm{C}_2 \sqrt{\frac{1}{s}\sum_{\nu=1}^{s} |P(x_\nu)|^2},$$

$\mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3$ – absolute positive constants.

Lunin points (1989) under the same conditions:

$\exists\, \Omega = \{x_1, \dots, x_s\} \subset X$, $s \leqslant \mathrm{C}_1 n$,

$$\mathrm{C}_3 \sqrt{\frac{1}{s}\sum_{\nu=1}^{s} |P(x_\nu)|^2} \leqslant \|P\|_{L^2}.$$

S. Nitzian, A. Olevskii, A. Ulanovskii

Proc. of the AMS, vol. **144**, No. 1, 109 – 118 (2016):

$\Lambda \subset \mathbb{R}$ – discrete set,

μ – Lebeasque measure on $\mathbb{R}$,

$$\mathrm{E}(\Lambda) = \{e^{i\lambda t}\}_{\lambda \in \Lambda}$$

Definition:

$\mathrm{E} = \{u_j\}_{j=1}^{\infty}$ is a frame in H, if there are positive constants a, A, such that

$$a\,\|h\|_H^2 \leqslant \sum_{u_j \in \mathrm{E}} |\langle h, u_j\rangle|^2 \leqslant A\,\|h\|_H^2 \quad \forall\ h \in H$$

THEOREM. *There are positive constants* c, C

such that for every set $S \subset \mathbb{R}, \quad \mu(S) < \infty$

there is a discrete set $\Lambda \subset \mathbb{R}$

such that $\mathrm{E}(\Lambda)$ *is a frame in* $L^2(S)$

with frame bounds $c\mu(S)$ *and* $C\mu(S)$.

Peter Oswald, Weiqi Zhou

"Random reordering in SOR methods"

Numer. Math., **135** (2017), 1207--1220

Srivastava reformulated the Theorem 1:

Let A – $N \times n$ matrix such that

for any $x \in \mathbb{R}^n$ and $i_0 \in \langle N \rangle$

$$|(v_{i_0}, x)| \leqslant \varepsilon \left(\sum_{i=1}^{N} |(v_i, x)|^2 \right)^{1/2}. \qquad (2)$$

Then there exists decomposition (1)

such that for any $x \in \mathbb{R}^n$

$$\sum_{i \in \Omega_k} |(v_i, x)|^p \leqslant \left(\frac{1}{2} + C\varepsilon \right) \sum_{i=1}^{N} |(v_i, x)|^p, \quad p = 2, \quad k = 1, 2. \qquad (3)$$

Srivastava asked if the similar result

holds true for $p = 1$:

"Is a statement like the above true for $p = 1$?"

Possible applications:

- Goddyn's conjecture on thin spanning trees
- Sharp imbedding of the subspace of L^1 in l_1^n

B. Kashin, I. Limonova

Math. Notes (*Zametki*),
vol. **106**, No. 1
(July 2019):

Decomposing a matrix into two submatrices with extremely small (2,1) norm

Proposition 1

$$If\ A - N \times n\ matrix\ with$$

$$\|A\|_{(2,1)} = 1, \qquad \|v_i\|_2 \leqslant \varepsilon, \quad i \in \langle N \rangle\,.$$

$$Then\ there\ exists\ decomposition\ (1)$$

$$with\ \big||\Omega_1| - |\Omega_2|\big| \leqslant 1\ such\ that$$

$$\big\|A(\Omega_k^0)\big\|_{(2,1)} \leqslant \frac{1}{\sqrt{2}} + 2\varepsilon, \qquad k = 1, 2.$$

Proposition 2

$$\textit{Let } n = 2^s, s = 1, 2, \ldots \textit{ and } n^{-1/2} \leqslant \varepsilon \leqslant 1.$$

$$\textit{There exists } 2n \times n \textit{ matrix } A = A(n, \varepsilon) \textit{ such that}$$

$$\textit{for any } x \in \mathbb{R}^n \textit{ and } i_0 \in \langle 2n \rangle$$

$$|(v_{i_0}, x)| \leqslant \varepsilon \sum_{i=1}^{N} |(v_i, x)|, \quad x \in \mathbb{R}^N, \quad i_0 \in \langle N \rangle, \qquad (4)$$

$$\textit{but for any decomposition } (1) \textit{ (with } N = 2n)$$

$$M \equiv \max(\|A(\Omega_1)\|_{(2,1)}, \|A(\Omega_2)\|_{(2,1)}) \geqslant$$
$$\geqslant \frac{1}{\sqrt{2}} \left(\frac{1}{1 + (\varepsilon n^{1/2})^{-1}} \right) \|A\|_{(2,1)}.$$

Proposition 3

Suppose that for $N \times n$ matrix A and some ε, $0 < \varepsilon \leqslant \frac{1}{n}$, the estimate (4) is true.

Then there exists decomposition (1) such that

$$\|A(\Omega_k)\|_{(2,1)} \leqslant \left(\frac{1}{2} + 2\varphi(n,\varepsilon)\right) \|A\|_{(2,1)}, \quad k = 1, 2,$$

where

$$\varphi(n,\varepsilon) = \left(n\varepsilon \ln \frac{8}{n\varepsilon}\right)^{1/3}.$$

Proposition 4

$$\textit{Suppose that for given } N \times n \textit{ matrix } A \textit{ and}$$
$$0 < \varepsilon \leqslant \frac{1}{n} \textit{ the estimate } (4) \textit{ is true}$$
$$\textit{and for any } x \in \mathbb{R}^n,\ x \neq 0$$

$$0 < b\|x\|_2 \leqslant \|Ax\|_1 \leqslant B\|x\|_2 .$$

$$\textit{Then there exists decomposition } (1) \textit{ such that}$$
$$\textit{for any } x \in \mathbb{R}^n \textit{ and } k = 1, 2$$

$$\|A(\Omega_k)x\|_1 \leqslant \gamma \|Ax\|_1,\ \gamma = \frac{1}{2} + 4\left(n\varepsilon \ln \frac{2B}{b\varepsilon^{1/3}n^{1/3}}\right)^{1/3} .$$

THEOREM 2. (J. Bourgain, 1989)

For any O.N.S. $\Phi = \{\varphi_i(x)\}_{i=1}^n$ *with*

$$\|\varphi_i\|_{L^\infty} \leqslant M, \quad i = 1, 2, \ldots, n, \qquad (*)$$

there exists $\sigma \in S(n)$ *such that*

$$\|S^*_{\Phi_\sigma} : l_2^n \to L^2\| \leqslant C_M \log\log n.$$

This is best possible result one may obtain by purely probabilistic method.

Some generalization: A. Lewko, M. Lewko "The square variation of rearranged Fourier series"

Amer. J. of Math.,

vol. **137**, No. 5, 1257 -- 1291 (2015)

THEOREM. (B. K., I. Limonova)

For any O.N.S. $\Phi = \{\varphi_i(x)\}_{i=1}^{n}$ *with the prorerty* $(*)$

and any $\gamma > 4$ *there exists* $\Lambda \subset \langle n \rangle$,

$|\Lambda| \geqslant n(\log n)^{-\gamma}$ *such that*

$$\|S^*_{\Phi_\Lambda} : l^\infty(\Lambda) \to L^2\| \leqslant C_{\gamma,M} |\Lambda|^{1/2},$$

where $\Phi_\Lambda = \{\varphi_i(x)\}_{i \in \Lambda}$.

Thank you for your attention !