On Hermite-Hadamard and Jensen inequalities

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Hermite-Hadamard inequalities

C. Hermite

J. Hadamard

Theorem 1 (Hermite 1881 & Hadamard 1893)

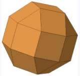
Let $f : \mathbb{R} \to \mathbb{R}$ concave. Then

$$\frac{f(-a)}{2} + \frac{f(a)}{2} \leq \frac{1}{2a} \int_{-a}^{a} f(x)dx \leq f\left(\frac{-a}{2} + \frac{a}{2}\right) = f(0).$$

S.S. Dragomir, C.E.M. Pearce, Selected Topics on Hermite-Hadamard Inequality and Applications

Hermite-Hadamard in $\mathbb{R}^n$

- $\mathcal{K}^n$ set of compact convex sets in $\mathbb{R}^n$.



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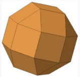

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$$x_K = \frac{1}{|K|}\int_K x dx.$$

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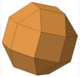

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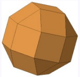

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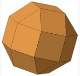

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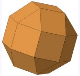

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Hermite-Hadamard for $f(x)^m$

Theorem 2 (Milman & Pajor '00)

Let $f : \mathbb{R}^n \to \mathbb{R}_+$ be s.t. $\log f$ is concave and $\mu : \mathbb{R}^n \to \mathbb{R}_+$ a probability measure. Then

$$\int_{\mathbb{R}^n} f(x) d\mu(x) \leq f\left(\int_{\mathbb{R}^n} x \frac{f(x)}{\int f(z) d\mu(z)} d\mu(x) \right)$$

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Corollary

Let $K \in \mathcal{K}^n$, $f : K \to \mathbb{R}_+$ concave, and $m \in \mathbb{N}$. Then

$$\frac{1}{|K|} \int_K f(x)^m dx \leq f(x_f)^m,$$

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Theorem 3 (G.M.+19, Dragomir '00)

Let $f : B_2^n \to \mathbb{R}_+$ concave and $m \in \mathbb{N}$. Then

$$\frac{1}{|B_2^n|}\int_{B_2^n} f(x)^m dx \leq \frac{2^{m+n}}{(m+n)!}\Gamma\left(\frac{2m+n+1}{2}\right)\Gamma\left(\frac{n+2}{2}\right) f(0)^m.$$

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Equality holds iff f is affine and if moreover $m \geq 2$ then K is a generalized cylinder s.t. $f \equiv 0$ in one of its basis.

Proof of Theorem 4

Proof. S1: replace f concave by $r : K \to [0, \infty)$ affine s.t.

$$r(0) = f(0) \quad \text{and} \quad r(x) \geq f(x) \,\forall\, x \in K,$$

and let $g(x) = r(x)/r(0)$.

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S3: After rotating, let $h(K, e_1) = t_0$ and

$$g(t, x_2, \ldots, x_n) = 1 + \frac{t}{t_0}\delta \quad \text{for every } t \in [-t_0, t_0].$$

where $\delta \in [0, 1]$.

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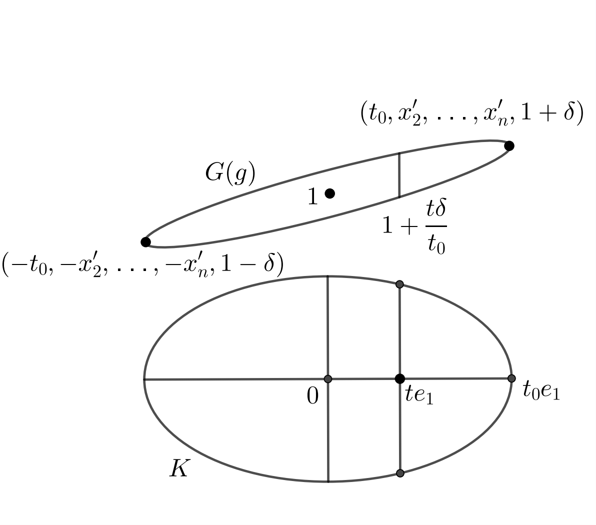

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$$R_t = (-te_1 + K_t') + [-t_0 e_1, t_0 e_1].$$

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S7: Since $R_{t_0} \subset K' \subset R_0$ and changes continuously on t, let

$$t^* \in [0, t_0] \quad \text{s.t.} \quad |R_{t^*}| = |K'|$$

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S8: Let $M''_t = R \cap (te_1 + e_1^\perp)$ and observe that

$$M''_t \subset M'_t \text{ if } t \in [0, t^*] \text{ and } M'_t \subset M''_t \text{ if } t \in [t^*, t_0].$$

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where $M^*_t = M'_t \cap M''_t$ and $M^{**}_t = (M'_t \setminus M''_t) \cup (M''_t \setminus M'_t)$.

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where $\varepsilon = 0$ if m is even, and $\varepsilon = 1$ if m is odd. Since that function is increasing, then

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$$= \int_{-t_0}^{-t^*}\left(1 + \frac{t}{t_0}\right)^m |M_t^{**}| dt + \int_{t^*}^{t_0}\left(1 + \frac{t}{t_0}\right)^m |M_t^{**}| dt.$$

Therefore we have proven that

$$\int_K \frac{f(x)^m}{f(0)^m} dx \leq \int_R g_0(x)^m dx,$$

where $g_0(x)$ is an affine function with $g_0(0) = 1$ and $g_0(-t_0, x_2, \ldots, x_n) = 0$ for every $(x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$,

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$$\int_K \frac{f(x)^m}{f(0)^m} dx \leq \frac{2^m}{m+1} |R|$$

Proof of Theorem 4

Therefore we have proven that

$$\int_K \frac{f(x)^m}{f(0)^m} dx \leq \int_R g_0(x)^m dx,$$

where $g_0(x)$ is an affine function with $g_0(0) = 1$ and $g_0(-t_0, x_2, \ldots, x_n) = 0$ for every $(x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$, and thus

$$\int_K \frac{f(x)^m}{f(0)^m} dx \leq \frac{2^m}{m+1} |R| = \frac{2^m}{m+1} |K|. \qquad \square$$

Reverse Hermite-Hadamard

Theorem 5

Let $0 \in K \in \mathcal{K}^n$, $f : K \to \mathbb{R}_+$ concave and $m \in \mathbb{N}$. Then

$$\binom{m+n}{n}^{-1} f(0)^m \leq \frac{1}{|K|}\int_K f(x)^m dx.$$

Theorem 5

Let $0 \in K \in \mathcal{K}^n$, $f : K \to \mathbb{R}_+$ concave and $m \in \mathbb{N}$. Then

$$\binom{m+n}{n}^{-1} f(0)^m \leq \frac{1}{|K|} \int_K f(x)^m dx.$$

Equality holds iff the graph of f is a cone with basis $K \times \{0\}$ and apex $(0, f(0))$.

Application

- $\mathcal{L}_i^n$ the set of i-dimensional linear subspaces in $\mathbb{R}^n$.

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Theorem 6 (Brunn 1887 & Minkowski 1896)

Let $K, C \in \mathcal{K}^n$. Then

$$|(1-\lambda)K + \lambda C|^{\frac{1}{n}} \geq (1-\lambda)|K|^{\frac{1}{n}} + \lambda |C|^{\frac{1}{n}}$$

for any $\lambda \in [0,1]$.

Application

Theorem 7 (Rogers & Shephard '58)

Let $K \in \mathcal{K}^n$ and $H \in \mathcal{L}_i^n$. Then

$$\binom{n}{i}^{-1} |P_H K| \cdot |K \cap H^{\perp}| \leq |K|.$$

Application

Theorem 7 (Rogers & Shephard '58)

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Theorem 8 (Fubini's formula)

Let $K \in \mathcal{K}^n$ and $H \in \mathcal{L}_i^n$. Then

$$|K| \leq |P_H K| \max_{x \in H} |K \cap (x + H^{\perp})|.$$

Application

Corollary (Spingarn '93, Milman & Pajor '00)

Let $K \in \mathcal{K}^n$ and $H \in \mathcal{L}_i^n$. Then

$$|K| \leq |P_H K| \cdot |K \cap (x_K + H^\perp)|.$$

Application

Corollary (Spingarn '93, Milman & Pajor '00)

Let $K \in \mathcal{K}^n$ and $H \in \mathcal{L}_i^n$. Then

$$|K| \leq |P_H K| \cdot |K \cap (x_K + H^\perp)|.$$

Corollary (Jensen 1906)

Let $K \in \mathcal{K}^n$ and $H \in \mathcal{L}_{n-1}^n$. Then

$$|K| \leq |P_H K| \cdot |K \cap (x_{P_H K} + H^\perp)|.$$

Application

Corollary (G.M.+19)

Let $K \in \mathcal{K}^n$ and $H \in \mathcal{L}_i^n$ with $P_H K = B_2^i$. Then

$$|K| \leq \frac{2^n}{\pi^{\frac{1}{2}} n!} \Gamma\left(\frac{2n-i+1}{2}\right) \Gamma\left(\frac{i+2}{2}\right) |P_H K| \cdot |K \cap H^{\perp}|.$$

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Corollary (G.M.+19)

Let $K \in \mathcal{K}^n$ and $H \in \mathcal{L}_i^n$ with $P_H K = -P_H K$. Then

$$|K| \leq \frac{2^{n-i}}{n-i+1} |P_H K| \cdot |K \cap H^\perp|.$$

Application

Proof of Corollary. By Fubini's formula

$$|K| = \int_{P_H K} |K \cap (x + H^{\perp})| dx.$$

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Proof of Corollary. By Fubini's formula

$$|K| = \int_{P_H K} |K \cap (x + H^\perp)| dx.$$

Let

$$f : P_H K \to \mathbb{R}_+ \quad \text{be s.t.} \quad f(x) = |K \cap (x + H^\perp)|^{\frac{1}{n-i}}.$$

Application

Proof of Corollary. By Fubini's formula

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$$\frac{|K|}{|P_H K|} = \frac{1}{|P_H K|} \int_{P_H K} f(x)^{n-i} dx$$

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$$\leq \frac{2^{n-i}}{n-i+1} f(0)^{n-i} = \frac{2^{n-i}}{n-i+1} |K \cap H^{\perp}|. \qquad \square$$

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Thank you for your attention!!