Geometry of spaces between polytopes and related zonotopes. *

Yehoram Gordon [†‡§] Alexander Litvak[‡ ¶]

Carsten Schütt[†] Elisabeth Werner [‖]

Abstract

We study geometric parameters associated with the Banach spaces $(\mathbb{R}^n, \|\cdot\|_{k,q})$ normed by $\|x\|_{k,q} = \left(\sum_{1\le i\le k} |\langle x, a_i\rangle|^{*q}\right)^{1/q}$, where $\{a_i\}_{i\le N}$ is a given sequence of N points in $\mathbb{R}^n$, $1 \le k \le N$, $1 \le q \le \infty$ and $\{\lambda_i^*\}_{i\ge 1}$ denotes the decreasing rearrangement of a sequence $\{\lambda_i\}_{i\ge 1} \subset \mathbb{R}$.

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0 Introduction

Let $a_i \in I\!R^n$, $i = 1, \ldots, N$ be a sequence of vectors such they span the space $I\!R^n$ and let

$$\|x\|_{k,q} = \left(\sum_{1 \le i \le k} |\langle x, a_i \rangle|^{*q} \right)^{\frac{1}{q}},$$

where $|\langle x, a_i \rangle|^*$, $i = 1, \ldots, N$ is the decreasing rearrangement of the sequence $|\langle x, a_i \rangle|$, $i = 1, \ldots, N$. We denote the normed space $(I\!R^n, \| \cdot \|_{k,q})$ by $X_{k,q}$. The unit ball of $\|x\|_{k,q}$ we denote by $B_{k,q}$. We investigate the geometry of these spaces and their duals in this paper.

The interest for those spaces comes from the fact that they generalize the class of dual spaces of zonotopes in a natural way. For $k = N$ and $q = 1$ the spaces $X^*_{k,q}$ are zonotopes and for $k = 1$ and $q = 1$ the spaces $X_{k,q}$ range over all possible spaces with a polytopal unit ball with no more than $2N$ facets.

The geometry of the spaces $X_{N,1}$ has been investigated in [GJ1], [GJ2], [GJN] while the spaces $X_{k,q}$ for arbitrary k and q were hardly considered in the literature.

We provide estimates for the volume of the unit balls of $X_{k,q}$ and their lower dimensional subspaces. We determine the dimension of almost Euclidean subspaces and thus obtain a Dvoretzky-type theorem.

In section 5 we investigate the special case when the set $\{a_i\}$ is $\{e_i\}$, the canonical basis of $I\!R^n$. Then the norm $\| \cdot \|_k = \| \cdot \|_{k,1}$ is in a sense intermediate between the ℓ_1-norm and the ℓ_∞-norm. Moreover, $(I\!R^n, \| \cdot \|_k)$ is an interpolation space between ℓ_1^n and ℓ_∞^n. In fact, by Lemma 5.1, $B_k = \text{conv}\, \{B_1^n, B_\infty^n / k\}$, where B_1^n and B_∞^n are the unit balls of ℓ_1^n and ℓ_∞^n respectively. We provide asymptotically sharp estimates of the most important parameters of those bodies such as type and cotype constants, p-summing norms, volume ratios, projection constants, etc. We would also like to note that the general case can be reduced to this special case. Indeed, let $\{a_i\}_{i \le N} \subset I\!R^n$ and $T \ : \ I\!R^N \longrightarrow I\!R^n$ be the linear operator defined by $Te_j = a_j$, $j \le N$. Considering the extreme points it is not hard to see that

$$B_{k,1} = \left(T \left((kB_1^N) \cap B_\infty^N \right) \right)^0 = T^{*\ -1} \left(\text{conv}\, \left\{ B_1^N, B_\infty^N / k \right\} \right).$$

So, if the properties of the operator T are known we can estimate parameters of $B_{k,1}$.

1 Definitions, notations, known results

We shall use the standard notation from the local theory of Banach spaces (see e.g. [MS1], [Pi1], [T]). Given a finite set $\mathcal{N}$, its cardinality is denoted by $|\mathcal{N}|$. We denote the canonical Euclidean norm on $I\!R^n$ by $| \cdot |$, the Euclidean unit ball by B_2^n, and the Euclidean unit sphere by S^{n-1}. The normalized Lebesgue measure on S^{n-1} will be denoted by $d\nu$ (or by $d\nu_{n-1}$ if we need to emphasize the dimension). By $\{e_i\}_{1\le i\le n}$ we denote the canonical basis of $\mathbb{R}^n$. The standard norm in ℓ_p^n, $p\ge 1$, is denoted by $|\cdot|_p$ and the unit ball of it is denoted by B_p^n.

Given $x\in\mathbb{R}$ by $[x]$ we denote the largest integer not exceeding x.

Given a sequence $\{\lambda_i\}_{i\le N}\subset\mathbb{R}$ by $\{\lambda_i^*\}_{i\le N}$ (resp. $\{|\lambda_i|^*\}_{i\le N}$) we denote the non-increasing rearrangement of $\{\lambda_i\}_{i\le N}$ (resp. $\{|\lambda_i|\}_{i\le N}$).

As mentioned in the introduction, given a sequence $\{a_i\}_{i\le N}\subset\mathbb{R}^n$ and $q\ge 1$ for every $k\le N$ we define the following norm on $\mathbb{R}^n$

$$\|x\|_{k,q}=\left(\sum_{i=1}^{k}\left(|\langle x,a_i\rangle|^*\right)^q\right)^{1/q}.$$

The unit ball of $\|x\|_{k,q}$ we denote by $B_{k,q}$. The norm $\|\cdot\|_{k,1}$ and its unit ball $B_{k,1}$ we denote by $\|\cdot\|_k$ and B_k. Let us note that for $q\ge\ln k$ one has

$$\max_{i\le N}|\langle x,a_i\rangle|=\|x\|_{1,1}\le\|x\|_{k,q}\le e\|x\|_{1,1}.$$

Therefore working with $\|x\|_{k,q}$ below we always assume that $q\le\ln k$.

By a convex body $K\subset\mathbb{R}^n$ we shall always mean a compact convex set with the non-empty interior, and without loss of generality we shall assume that interior of K contains 0. The gauge of K is denoted by $\|\cdot\|_K$, i.e., $\|x\|_K=\inf\{\lambda>0\mid x\in\lambda K\}$. The n-dimensional volume of K is denoted by $|K|$.

The n-dimensional normed space defined by a norm $\|\cdot\|$ (resp. by a centrally-symmetric convex body K) we denote by $(\mathbb{R}^n,\|\cdot\|)$ (resp. $(\mathbb{R}^n,K)$). Usually we identify the n-dimensional normed space with its unit ball.

Given centrally-symmetric convex bodies K, L in $\mathbb{R}^n$, we define the Banach–Mazur distance by

$$d(K,L)=\inf\{\alpha\beta\mid\alpha>0,\beta>0,(1/\beta)L\subset UK\subset\alpha L\},$$

where the infimum is taken over all linear $U:\mathbb{R}^n\to\mathbb{R}^n$.

By $\{g_i\}$, $\{h_i\}$, $\{g_{i,j}\}$ we shall always denote sequences of independent standard Gaussian random variables. Given integers m, n by the Gaussian operator $G\ :\ \mathbb{R}^m\longrightarrow\mathbb{R}^n$ we mean the operator

$$G=\sum_{i\le m,j\le n}g_{i,j}e_i\otimes e_j. \tag{1}$$

By g we denote the standard Gaussian vector in $\mathbb{R}^n$, i.e. $g=\sum_{i=1}^n g_ie_i$. The expectation of the Gaussian vector in the space $X=(\mathbb{R}^n,K)$ is denoted by

$$E(X)=E(K):=\mathbf{E}\|g\|_K. \tag{2}$$

It is well known (and can be directly checked) that

$$E(X) \leq \sqrt{n} \int_{S^{n-1}} \|x\| d\nu \leq c_n E(X),$$

where c_n tends to 1 as n grows to infinity. We also denote

$$\varepsilon_2(K) = \|Id \ : \ \ell_2^n \longrightarrow (\mathbb{R}^n, K)\| = \max_{\sum t_i^2 = 1} \left\{ \left\| \sum_{i \leq n} t_i e_i \right\|_K \right\}. \tag{3}$$

Given two sequences $a = \{a_i\}$ and $b = \{b_i\}$ by $a \cdot b$ we denote the sequence $\{a_i b_i\} = \{(a \cdot b)_i\}$.

We recall also the definitions of an Orlicz function and an Orlicz norm. A convex function $M : \mathbb{R}^+ \to \mathbb{R}^+$ with $M(0) = 0$ and $M(t) > 0$ for $t \neq 0$ is called an Orlicz function. The Orlicz norm on $\mathbb{R}^n$ is defined by

$$\|x\|_M = \inf \left\{ \rho > 0 \ : \ \sum_{i=1}^n M\left(|x_i|/\rho\right) \leq 1 \right\}.$$

Any Orlicz function M can be represented as

$$M(t) = \int_0^t p(s) ds,$$

where $p(t)$ is a non-decreasing, right continuous function. If $p(t)$ satisfies

$$p(0) = 0 \qquad \text{and} \qquad p(\infty) = \lim_{t \to \infty} p(t) = \infty, \tag{4}$$

we define the dual Orlicz function M^* by

$$M^*(t) = \int_0^t q(s) ds,$$

where $q(s) = \sup\{t \ : \ p(t) \leq s\}$. Such a function M^* is also an Orlicz function and

$$\|x\|_M \leq |||x||| \leq 2\|x\|_M,$$

where $||| \cdot |||$ is the dual norm to $\| \cdot \|_{M^*}$ (see e.g. [LT]). Moreover,

$$s < M^{*-1}(s) M^{-1}(s) \leq 2s$$

for every positive s (see e.g. 2.10 of [KR]). The last inequality shows in particular that to define an Orlicz norm $\| \cdot \|_M$ it is enough to define the function M^{*-1}. We shall use this below. Note that the condition (4) in fact excludes only the case $M(t)$ is equivalent to t, i.e. the case when there are absolute positive constants c, C such that $ct \leq M(t) \leq Ct$. Moreover, q satisfies condition (4) as well and $q = p^{-1}$ if p is an invertible function. We refer to [KR, LT] for further properties of Orlicz functions.

The letters C, c, c_0, c_1, ... denote absolute positive constants whose values may be different from line to line.

2 Preliminary results

The following lemma can be proved by direct computations.

Lemma 2.1 *Let N be an integer. Consider the sequence $\{|g_i|\}_{i\le N}$. For every $k \le N/2$ one has*

$$c\sqrt{\ln(3N/k)} \le \ \mathbf{E}|g_k|^* \le C\sqrt{\ln(3N/k)},$$

where c, C are absolute constants.

Remark. Thus for every $k \le N$ one has

$$ck\sqrt{\ln(3N/k)} \le \ \mathbf{E}\sum_{i=1}^{k}|g_i|^* \le Ck\sqrt{\ln(3N/k)},$$

where c, C are absolute constants.

Throughout we shall use the following inequality proved in [Go3].

Theorem 2.2 *Let $\{X_i\}_{i\le N}$ and $\{Y_i\}_{i\le N}$ be two sequences of centered Gaussian random variables which satisfy*

$$\mathbf{E}\left|X_i - X_j\right|^2 \le \mathbf{E}\left|Y_i - Y_j\right|^2$$

for all i, j. Then for all $k \le N$ we have

$$\mathbf{E}\sum_{i=1}^{k} X_i^* \le \mathbf{E}\sum_{i=1}^{k} Y_i^*.$$

As a corollary we have

Lemma 2.3 *Let $\{a_j\}_{j\le N} \subset I\!R^n$. Then for every $k \le N$*

$$c\varepsilon k\sqrt{\ln(3N/k)} \le E(B_k),$$

where $\varepsilon = \min_{i\ne j}|a_i - a_j|$ and $c > 0$ is an absolute constant.

Moreover, if $\{a_j\}_{j\le 2N} \subset S^{n-1}$ then for every $k \le N$

$$E(B_k) \le Ck\sqrt{\ln(3N/k)},$$

where C is an absolute constant.

Proof: First we show the "Moreover" part of the lemma. Let g be the standard Gaussian vector in $\mathbb{R}^n$. Define centered Gaussian random variables by $X_i = \langle g, a_i \rangle$, $Y_i = \sqrt{2} h_i$, for $i \leq N$, and $X_i = -X_{i-N}$, $Y_i = -Y_{i-N}$ for $N < i \leq 2N$. Then

$$\mathbf{E} \left| X_i - X_j \right|^2 \leq 4 \leq \mathbf{E} \left| Y_i - Y_j \right|^2 .$$

Since for every $k \leq N$

$$\sum_{i=1}^k |X_i|^* = \sum_{i=1}^k X_i^* \quad \text{and} \quad \sum_{i=1}^k |Y_i|^* = \sum_{i=1}^k Y_i^*,$$

using the previous two statements, we obtain

$$E(B_k) = \mathbf{E} \sum_{i=1}^k X_i^* \leq \mathbf{E} \sum_{i=1}^k Y_i^* \leq ck \sqrt{\ln(3N/k)},$$

which proves the upper estimate.

Let us turn to the lower estimate. Let X_i, $1 \leq i \leq 2N$, be as above and define now centered Gaussian random variables Y_i by $Y_i = \varepsilon h_i/2$, $i \leq N$, $Y_i = -Y_{i-N}$, $N < i \leq 2N$.

Then by Theorem 2.2 and Lemma 2.1 we obtain

$$E(B_k) = \mathbf{E} \sum_{i=1}^k X_i^* \geq \mathbf{E} \sum_{i=1}^k Y_i^* = \mathbf{E} \sum_{i=1}^k |Y_i|^* \geq c\varepsilon k \sqrt{\ln(3N/k)},$$

which proves the lemma. □

We shall need the following theorem from [GLSW] (Theorem 4 with the remark after the proof of Proposition 6 and Example 16).

Theorem 2.4 *Let $k \leq N$ and $1 \leq q \leq \ln N$. Let $\lambda = \{\lambda_i\}_{i \leq N} \subset \mathbb{R}$. Let $\bar{g} = \{|g_i|^q\}_{i \leq N}$, $\bar{f} = \{|f_i|^q\}_{i \leq N}$, where $\{g_i\}_{i \leq N}$ denotes a sequence of independent standard Gaussian random variables and $\{f_i\}_{i \leq N}$ denotes a sequence of standard Gaussian random variables (not necessarily independent). Then*

$$\mathbf{E} \sum_{i=1}^k \left| \left(\lambda \cdot \bar{f} \right)_i \right|^* \leq \frac{4e}{e-1} \, \mathbf{E} \sum_{i=1}^k \left| (\lambda \cdot \bar{g})_i \right|^*$$

and

$$(cq)^{q/2} \, \|\lambda\|_{M_{k,q}} \leq \; \mathbf{E} \sum_{i=1}^k \left| (\lambda \cdot \bar{g})_i \right|^* \leq (Cq)^{q/2} \, \|\lambda\|_{M_{k,q}}$$

where $0 < c < 1 < C$ are absolute constants and $M_{k,q}$ is the Orlicz function defined by

$$M_{k,q}(t) = \begin{cases} 0 & t = 0 \\ \frac{1}{k}\exp\left(-q/\left(kt\right)^{2/q}\right) & t \in (0, t_0) \\ at - b & t \geq t_0, \end{cases}$$

$$t_0 = \frac{1}{k}\left(\frac{2q}{q+2}\right)^{q/2}, \qquad a = \frac{q+2}{eqkt_0}e^{-q/2}, \qquad b = \frac{2}{eqk}e^{-q/2}.$$

Remark 1. Note that the inequality in the remark after Lemma 2.1 follows from this theorem as well.

Remark 2. Let us mention that to prove the theorem we use that for every Orlicz function M there exists a sequence $y_1 \geq y_2 \geq ... \geq y_n > 0$ such that

$$\frac{e-1}{2e}\|x\|_M \leq n^{-n+1} \sum_{1 \leq j_1, \ldots, j_n \leq n} \max_{1 \leq i \leq n} |x_i y_{j_i}| \leq 2\|x\|_M$$

(see Lemma 5 and Lemma 9 of [GLSW]). We would like to note also that all ℓ_p-norms are Orlicz norms with the Orlicz function $M(t) = |t|^p$.

The theorem leads to the following extensions of Lemma 2.3.

Corollary 2.5 *Let $\{a_j\}_{j \leq N} \subset I\!R^n$. Then for every $k \leq N$*

$$E(B_k) \leq C\, \|\{|a_i|\}\|_{M_{k,1}},$$

where C is an absolute constant and $M_{k,1}$ as in the previous theorem.

Moreover, denoting $\lambda_i = \min_{j \neq i} |a_i - a_j|$, we have

$$c\, \|\{\lambda_i\}\|_{M_{k,1}} \leq E(B_k)$$

for some absolute constant $c > 0$.

Proof: The proof mimics the proof of Lemma 2.3. Indeed, to obtain the upper estimate we need to define $X_i = \langle g, a_i \rangle$, $X_{N+i} = -X_i$, and $Y_i = \sqrt{2}|a_i|h_i$, $Y_{N+i} = -Y_i$, for every $i \leq N$. To obtain the lower estimate we take the same X_i and $Y_i = \lambda_i h_i/2$, $i \leq N$, $Y_i = -Y_{i-N}$, $N < i \leq 2N$. $\square$

Remark. It can be shown that

$$\|\{\lambda_i\}\|_{M_{k,1}} \approx \sum_{i=1}^{k} |\lambda_i|^* + k \max_{i \leq N/k} \lambda_{ki}^* \sqrt{1 + \ln i}$$

for every $\lambda \in I\!R^n$.

Corollary 2.6 *Let $q \geq 1$. Let $\{a_j\}_{j \leq N} \subset \mathbb{R}^n$. Then for every $k \leq N$*

$$E(B_{k,q}) \leq C\sqrt{q}\left(\|\{|a_i|^q\}\|_{M_{k,q}}\right)^{1/q},$$

where C is an absolute constant and $M_{k,q}$ as in the previous theorem.

Proof: We apply Theorem 2.4 to the standard Gaussian random variables $f_i = \langle g, a_i\rangle/|a_i|$, where g is the standard Gaussian vector in $\mathbb{R}^n$. Let $\bar{f} = \{|f_i|^q\}_{i \leq N}$ and $\lambda = \{|a_i|^q\}_{i \leq N}$. We obtain

$$E(B_{k,q}) = \mathbf{E}\left(\sum_{i=1}^{k}(|\langle g, a_i\rangle|^*)^q\right)^{1/q} \leq \left(\mathbf{E}\sum_{i=1}^{k}(|\langle g, a_i\rangle|^*)^q\right)^{1/q} =$$

$$\left(\mathbf{E}\sum_{i=1}^{k}\left|(\lambda\cdot\bar{f})_i\right|^*\right)^{1/q} \leq \left(\frac{4e}{e-1}\,\mathbf{E}\sum_{i=1}^{k}|(\lambda\cdot\bar{g})_i|^*\right)^{1/q} \leq$$

$$C\sqrt{q}\left(\|\{|a_i|^q\}\|_{M_{k,q}}\right)^{1/q}.$$

□

We conclude this section with the piecewise continuous version of theorem 2.4, namely

Corollary 2.7 *If $\{a_\tau, 0 \leq \tau \leq 1\}$ is a piecewise continuous path in $\mathbb{R}^n$, and if $0 < t \leq 1$, $q \geq 1$ are fixed, and if*

$$\sup|a_\tau|^q \leq \left(\frac{2q}{q+2}\right)^{q/2} x_0,$$

where

$$x_0 := \inf\left\{x > 0 \ : \ \int_0^1 \exp\left(\frac{-qx^{2/q}}{|a_\tau|^2}\right)d\tau \leq t\right\},$$

then

$$E\left(\frac{1}{t}\int_0^t |< g, a_\tau >|^{*q}d\tau\right) \leq (cq)^{q/2}x_0.$$

Remark. If $a_\tau \subset S^{n-1}$ then it follows that $x_0 = \left(\frac{1}{q}\log\left(\frac{1}{t}\right)\right)^{q/2}$ and the condition is that t satisfies $0 < t \leq e^{-\frac{q+2}{2}}$. A more careful analysis of the discrete version can give an estimate valid for all piecewise continuous paths, that will hold for all values of t.

3 Volume estimates

In this section we shall obtain two-sided estimates on the volumes of the bodies $B_{k,q}$ and their l-dimensional sections. We start with the following theorem, which can be obtained also as a corollary to the general result proved in Theorem 3.2.

Theorem 3.1 *Let $n \le N$ be positive integers. Let $\{a_i\}_{i \le N} \subset S^{n-1}$. Then for every $k \le N$ one has*

$$|B_k|^{1/n} \ge \frac{C}{k\sqrt{\ln \frac{3N}{k+n}}},$$

where $C > 0$ is an absolute constant.

Remark. Let $k = 1$. Then $B_1^0 = \text{conv}\,\{a_i\}$, and

$$|B_1|^{1/n} \ge \frac{C}{\sqrt{\ln \frac{3N}{n}}},$$

which was proved independently by Bárány and Füredy ([BF]), Carl and Pajor ([CP]) and Gluskin ([G1]). See also [FJ] and Corollary 2.2 of [GJ1], which generalizes it for an arbitrary set $\{a_i\} \subset I\!R^n$. Thus our Theorem 3.1 extends this result.

Proof: Let g be the standard Gaussian vector. By integration over the Euclidean sphere and Lemma 2.3, we obtain

$$\begin{aligned}
\left(|B_k| \,/\, |B_2^n|\right)^{1/n} &= \left(\int_{S^{n-1}} \|x\|_k^{-n}\, d\nu(x)\right)^{1/n} \\
&\ge \left(\int_{S^{n-1}} \|x\|_k\, d\nu(x)\right)^{-1} \\
&\ge c_1\sqrt{n}/\ \mathbf{E}\,\|g\|_k \\
&= c_1\sqrt{n}/\sum_{i=1}^{k} \mathbf{E}\ |\langle g, a_i\rangle|^* \\
&\ge \frac{c_2\sqrt{n}}{k\sqrt{\ln\left(3N/k\right)}},
\end{aligned}$$

where c_1 and c_2 are absolute positive constants.

To conclude the proof it is enough to notice that $\|x\|_k \le k\|x\|_1$. Together with the result of the remark above this implies that

$$|B_k|^{1/n} \ge \frac{1}{k}\,|B_1|^{1/n} \ge \frac{c_2}{k\sqrt{\ln \frac{3N}{n}}},$$

where c_2 is an absolute positive constant and the theorem follows. $\square$

Remark. Since

$$\left(\left|B_k^0\right| / \left|B_2^n\right|\right)^{1/n} \leq \left(\left|\partial B_k^0\right| / \left|\partial B_2^n\right|\right)^{1/(n-1)} \leq c_1 \int_{S^{n-1}} \|x\|_k \, d\nu(x),$$

using Urysohn's inequality, one can similarly show that

$$\left|\partial B_k^0\right|^{1/(n-1)} \leq \frac{ck}{n} \sqrt{\ln\left(3N/k\right)}.$$

The following theorem generalizes Theorem 3.1.

Theorem 3.2 *Let $n \leq N$ be positive integers. Let $q \geq 1$ and $\{a_i\}_{i \leq N} \subset I\!\!R^n$.*

(i) For every $k \leq N$ and every $l \leq n$ there exists an l-dimensional subspace $E \subset I\!\!R^n$ such that

$$|B_{k,q} \cap E|^{1/l} \;\geq\; C \; \frac{\sqrt{n}}{\sqrt{lq} \;\; \left(\| \; \{|a_i|^q\}_{i \leq N} \|_{M_{k,q}}\right)^{1/q}},$$

where $\| \cdot \|_{M_{k,q}}$ is the Orlicz norm with the function $M_{k,q}(t)$ defined in Theorem 2.4 and $C > 0$ is an absolute constant.

(ii) For every $k \leq N$ and every l-dimensional subspace $E \subset I\!\!R^n$ we have

$$|B_{k,q} \cap E|^{1/l} \geq$$

$$\frac{c}{\max_{|I|=l} \; |\det \; (Q_E a_i)_{i \in I} \; |^{1/l}} \left(\frac{1}{k^{\frac{1}{q}} \; \sqrt{\ln(3N/l)}} + \frac{1}{(\sqrt{q} \wedge \sqrt{l}) \; N^{\frac{1}{q}}} \right),$$

where $Q_E : I\!\!R^n \to E$ is the orthogonal projection onto E and $c > 0$ is an absolute constant.

(iii) For every $k \leq N$ and every l-dimensional subspace $E \subset I\!\!R^n$ we have

$$|B_{k,q} \cap E|^{1/l} \;\leq\; C \; \frac{N^{\frac{1}{q}}}{(l \;\, k)^{\frac{1}{q}} \; \left(\sum_{|I|=l} \; |\det \; (Q_E a_i)_{i \in I} \; |^q \right)^{\frac{1}{lq}}}$$

where $Q_E : I\!\!R^n \to E$ is the orthogonal projection onto E and $C > 0$ is an absolute constant.

Remark. Notice that if $\{a_i\}_{i\le N}\subset S^{n-1}$ then

$$\| \{|a_i|^q\}_{i\le N}\|_{M_{k,q}}^{1/q} \approx k^{1/q} \ \frac{\sqrt{\ln(3N/k)}}{\sqrt{q}}$$

and

$$|\det\ (Q_E a_i)_{i\in I}\ | \le 1.$$

Thus in this case, for every l there exists an l-dimensional subspace E such that

$$|B_{k,q}\cap E|^{1/l} \ge C\ \frac{\sqrt{n}}{k^{\frac{1}{q}}\ \sqrt{l}\ \sqrt{\ln(3N/k)}}.$$

Moreover, in this case for all l-dimensional subspaces E we have

$$|B_{k,q}\cap E|^{1/l} \ge c\ \left(\frac{1}{k^{\frac{1}{q}}\ \sqrt{\ln(3N/l)}} + \frac{1}{(\sqrt{q}\wedge\sqrt{l})\ N^{\frac{1}{q}}}\right).$$

If in addition $q=1$ in the last expression, then

$$|B_{k,1}\cap E|^{1/l} \ge c\ \left(\frac{1}{k\ \sqrt{\ln(3N/l)}} + \frac{1}{N}\right).$$

Proof of Theorem 3.2:

(i) Let $G_{n,m}$ denote the Grassmanian of m-dimensional subspaces of $I\!R^n$ and $d\mu$ denote the normalized Haar measure on it. Then integration over the Grassmanian gives

$$\begin{aligned}
\max_{H\subset G_{n,l}}\left(\frac{|H\cap B_{k,q}|}{|B_2^l|}\right)^{\frac{1}{l}} &\ge \left(\int_{G_{nl}} \frac{|H\cap B_{k,q}|}{|B_2^l|}\ d\mu(H)\right)^{1/l} \\
&= \left(\int_{S^{n-1}} \|x\|_{k,q}^{-l}\ d\nu(x)\right)^{1/l} \\
&\ge \left(\int_{S^{n-1}} \|x\|_{k,q}\ d\nu(x)\right)^{-1} \\
&\ge \frac{c\ \sqrt{n}}{E(B_{k,q})} \\
&\ge \frac{c_1\ \sqrt{n}}{\sqrt{q}\ \left(\|\ \{|a_i|^q\}_{i\le N}\|_{M_{k,q}}\right)^{1/q}},
\end{aligned}$$

where the last inequality follows by Corollary 2.6. That proves (i).

(ii) We first give a proof for the first expression.

As $B_{k,q} \supset \frac{1}{k^{1/q}} B_1$, it is enough to consider $B_1 \cap E$. Without loss of generality assume that $\{a_i\}$ is symmetric. By the inverse Santaló inequality [BM]

$$|B_1 \cap E|^{1/l} \geq \frac{c}{l\,|(B_1 \cap E)^0|^{1/l}}$$

and therefore it is enough to estimate the volume of the polar of the section $B_1 \cap E$ from above. The polar of $B_1 \cap E$ is the orthogonal projection of the polar $B_1^0 = \text{conv}\,\{\pm a_i : 1 \leq i \leq N\}$ onto E. Observe that

$$\text{conv}\,\{\pm a_i : 1 \leq i \leq N\} = T(B_1^N),$$

where B_1^N is the N-dimensional l_1 unit ball and $T : I\!R^N \to I\!R^n$ is the map defined by $T(e_i) = a_i, 1 \leq i \leq N$.

By a result of Meyer and Pajor [MP]

$$\frac{|T(B_1^N)|^{\frac{1}{n}}}{|B_1^n|^{\frac{1}{n}}} \leq \frac{|T(B_p^N)|^{\frac{1}{n}}}{|B_p^n|^{\frac{1}{n}}}$$

for all $1 \leq p < \infty$ and by a result of Gordon and Junge [GJ1] we have for all p' with $p' = \frac{p}{p-1}$

$$\frac{|T(B_p^N)|^{\frac{1}{n}}}{|B_p^n|^{\frac{1}{n}}} \leq c\sqrt{p'} \left(\sum_{|I|=n} |\det\,(a_i)_{i\in I}|^{p'} \right)^{\frac{1}{np'}},$$

where c is a constant. Therefore

$$\frac{|T(B_1^N)|^{\frac{1}{n}}}{|B_1^n|^{\frac{1}{n}}} \leq c\sqrt{p'} \left(\binom{N}{n} \max_{|I|=n} |\det\,(a_i)_{i\in I}|^{p'} \right)^{\frac{1}{np'}}$$

and thus

$$|T(B_1^N)|^{1/n} \leq c\sqrt{p'} \left(\frac{Ne}{n} \right)^{1/p'} \max_{|I|=n} |\det\,(a_i)_{i\in I}|^{1/n}\, |B_1^n|^{1/n}.$$

Hence for $|(B_1 \cap E)^0|^{1/l} = |Q_E(T(B_1^N))|^{1/l}$ where $Q_E : I\!R^n \longrightarrow E$ is the orthogonal projection, we get

$$|Q_E(T(B_1^N))|^{1/l} \leq c\sqrt{p'} \left(\frac{Ne}{l} \right)^{1/p'} \max_{|I|=l} |\det\,(Q_E a_i)_{i\in I}|^{1/l}\, |B_1^l|^{1/l}.$$

We choose $p' = \ln\frac{Ne}{l}$ so that $\sqrt{p'}\left(\frac{Ne}{l}\right)^{1/p'}$ is minimal and then observe that

$$|Q_E(T(B_1^N))|^{1/l} \leq \frac{c}{l}\sqrt{\ln\frac{Ne}{l}} \max_{|I|=l} |\det\,(Q_E a_i)_{i\in I}|^{1/l}.$$

Since $(B_1 \cap E)^0 = Q_E T B_1^N$, we obtain

$$|B_1 \cap E|^{1/l} \geq \frac{c}{\sqrt{\ln \frac{Ne}{l}} \ \max_{|I|=l} |\det\ (Q_E a_i)_{i\in I}|^{1/l}}.$$

Therefore

$$|B_{k,q} \cap E|^{1/l} \geq \frac{c}{k^{\frac{1}{q}} \ \sqrt{\ln \frac{Ne}{l}} \ \max_{|I|=l} |\det\ (Q_E a_i)_{i\in I}|^{1/l}}.$$

Now we give a proof for the second expression.

Note that

$$B_{N,q} \subset B_{k,q} \subset \left(\frac{N}{k}\right)^{\frac{1}{q}} B_{N,q} \tag{5}$$

and

$$B_{N,q}^0 = T(B_p^N), \tag{6}$$

where B_p^N is the unit ball of l_p^N, $\frac{1}{p} + \frac{1}{q} = 1$, and $T : I\!R^N \to I\!R^n$ is the map defined by $T(e_i) = a_i, 1 \leq i \leq N$.

By the inverse Santaló inequality [BM] we have for every l-dimensional subspace E

$$|B_{k,q} \cap E|^{1/l} \geq \frac{c}{l\ |(B_{N,q} \cap E)^0|^{1/l}}$$

and therefore it is enough to estimate the volume of the polar of the section $B_{N,q} \cap E$ from above. Again, the polar of the section $B_{N,q} \cap E$ is the orthogonal projection of the polar $B_{N,q}^0 = T(B_p^N)$ onto E.

By [GJ1]

$$c_2 \ \left(\sum_{|I|=n} |\det\ (a_i)_{i\in I}|^q\right)^{\frac{1}{nq}} \leq$$

$$\frac{|T(B_p^N)|^{\frac{1}{n}}}{|B_p^n|^{\frac{1}{n}}} \leq c_1\ (\sqrt{q} \wedge \sqrt{n}\ ) \left(\sum_{|I|=n} |\det\ (a_i)_{i\in I}|^q\right)^{\frac{1}{nq}}, \tag{7}$$

where c_1 and c_2 are constants. Therefore

$$\frac{|T(B_p^N)|^{\frac{1}{n}}}{|B_p^n|^{\frac{1}{n}}} \leq c_1\ (\sqrt{q} \wedge \sqrt{n}\ ) \left(\binom{N}{n} \max_{|I|=n} |\det\ (a_i)_{i\in I}|^q\right)^{\frac{1}{nq}}$$

and thus

$$|T(B_p^N)|^{1/n} \leq c_1\ (\sqrt{q} \wedge \sqrt{n}\ ) \left(\frac{Ne}{n}\right)^{1/q} \ \max_{|I|=n} |\det\ (a_i)_{i\in I}|^{1/n}\ |B_p^n|^{1/n}.$$

Hence for $|(B_{N,q} \cap E)^0|^{1/l} = |Q_E(T(B_p^N))|^{1/l}$ with the orthogonal projection $Q_E : I\!R^n \longrightarrow E$ we get

$$|Q_E(T(B_p^N))|^{1/l} \le c\ (\sqrt{q} \wedge \sqrt{l}\ ) \left(\frac{Ne}{l}\right)^{1/q}\ \max_{|I|=l}\ |\det\ (Q_E a_i)_{i\in I}|^{1/l}\ |B_p^l|^{1/l}$$

$$\le C\ (\sqrt{q} \wedge \sqrt{l}\ )\ \frac{N^{1/q}}{l}\ \max_{|I|=l}\ |\det\ (Q_E a_i)_{i\in I}|^{1/l}.$$

Therefore

$$|B_{k,q} \cap E|^{1/l} \ge \frac{c}{(\sqrt{q} \wedge \sqrt{l})\ N^{\frac{1}{q}}\ \max_{|I|=l}\ |\det\ (Q_E a_i)_{i\in I}|^{1/l}}.$$

(iii) By (5) and Santaló inequality we have for all l-dimensional subspaces E

$$|B_{k,q} \cap E|^{1/l} \le \frac{C}{l} \left(\frac{N}{k}\right)^{\frac{1}{q}} \frac{1}{|B_{N,q}^0 \cap E|^{1/l}},$$

which by (7) is

$$\le C\ \frac{N^{\frac{1}{q}}}{(k\ l)^{\frac{1}{q}}\ \left(\sum_{|I|=l} |\det\ (Q_E a_i)_{i\in I}|^q\right)^{\frac{1}{lq}}}.$$

□

Theorem 3.3 *Let k, n, N be integers such that $20^4 nk \le N \le 20^n nk$. There exists a sequence $\{a_i\}_{i\le N} \subset S^{n-1}$ such that for every $l \le n$ and every l-dimensional subspace $E \subset I\!R^n$ one has*

$$|B_{k,q} \cap E|^{1/l} \le C \frac{\sqrt{n}}{k^{\frac{1}{q}}\ \sqrt{l}\sqrt{\ln(N/(nk))}}.$$

Remark. Clearly, $(1/k)B_2^n \subset B_{k,q}$. On the other hand, if $N > k5^n$ then we can take k copies of some $1/2$-net in S^{n-1} (i.e. a sequence $\{a_{ij}\}_{i\le k, j\le 5^n}$, where $\{a_{ij}\}_{j\le 5^n}$ is the same $1/2$-net for each fixed i). Then for all i

$$\|x\|_{k,q} \ge k\ \max_j\ |\langle a_{ij}, x\rangle| \ge \frac{3}{4}\ k\ |x|.$$

Thus $B_{k,q} \subset \frac{4}{3k}\ B_2^n$ and, hence, for some positive absolute constants c, C

$$c\frac{1}{k\ \sqrt{l}}\ \le\ |B_{k,q} \cap E|^{1/l} \le\ C\frac{1}{k\ \sqrt{l}}.$$

We shall need the following simple fact. Let ρ be the geodesic distance on the Euclidean sphere S^{n-1}. Let $S(x,\delta)$ denote the cap with center x and radius δ

$$S(x,\delta)=\left\{y\in S^{n-1}\mid \rho(x,y)\le\delta\right\}.$$

When the choice of the center is not important we write just $S(\delta)$. As before $\nu=\nu_{n-1}$ denotes the normalized Lebesgue measure on S^{n-1}.

Fact 3.4 *For every $\delta\in[0,\pi/2]$ and $n\ge 3$ one has*

$$\frac{\delta\sin^{n-2}\delta}{2e(n-1)I}\le \nu(S(\delta))=\frac{1}{2I}\int_{\pi/2-\delta}^{\pi/2}\cos^{n-2}t\ dt\le\frac{\delta\sin^{n-2}\delta}{2I},$$

where

$$\frac{1}{\sqrt{n-1}}\le I=\int_0^{\pi/2}\cos^{n-2}t\ dt\le\sqrt{\frac{\pi}{2(n-1)}}.$$

Proof: The equality for $\nu(S(\delta))$ follows from the direct computation as well as the inequalities for I (see e.g. [MS1], Ch. 2). The upper inequality for $\nu(S(\delta))$ is obvious, since the cos is a decreasing function on $[0,\pi/2]$. Now let $\beta\in(0,1)$. Then

$$I_0:=\int_{\pi/2-\delta}^{\pi/2}\cos^{n-2}t\ dt\ge\int_{\pi/2-\delta}^{\pi/2-\beta\delta}\cos^{n-2}t\ dt\ge$$

$$(1-\beta)\delta\sin^{n-2}(\beta\delta)\ge(1-\beta)\beta^{n-2}\delta\sin^{n-2}\delta,$$

since $\sin(\beta\delta)\ge\beta\sin\delta$ for $\beta\in[0,1]$, $\delta\in[0,\pi/2]$. Taking $\beta=(n-2)/(n-1)$ we obtain $I_0\ge\frac{1}{e(n-1)}\delta\sin^{n-2}\delta$. That concludes the proof. $\Box$

Proof of Theorem 3.3: Our proof based on a construction by Figiel and Johnson ([FJ], see also [G1]).

It is enough to show the result for B_k as $B_{k,q}\subset k^{1-\frac{1}{q}}\ B_k$.

Take $m=[\log_{20}(N/(nk))]$. Then $4\le m\le n$. Denote $M=2[Nm/n]$ and choose δ such that $M=(\pi/\delta)^{m-1}$, i.e. $\delta=2\pi(1/M)^{1/(m-1)}<\pi/10$.

The standard volume estimates show that there exists a symmetric sequence $\{z_i\}_{i\le M}$, i.e. $\{-z_i\}_i=\{z_i\}_i$, which is a δ-net (with respect to geodesic distance) in S^{m-1}. Indeed, take a maximal δ-separated set $\mathcal{N}$ on the sphere. Clearly, $\mathcal{N}\cup-\mathcal{N}$ is a symmetric δ-net on S^{m-1}. Let a be the cardinality of $\mathcal{N}$. Since the caps $S(\delta/2)$ with the centers in $\mathcal{N}$ are disjoint we have $a\nu(S(\delta/2))\le\nu(S^{m-1})=1$. Using Fact 3.4 and the inequality $\sin\delta\ge 2\sqrt{2}\delta/\pi$ on $[0,\pi/4]$, we obtain for $m\ge 12$

$$a\le\sqrt{\frac{2\pi}{m-1}}\ \frac{e(m-1)}{\delta/2\sin^{m-2}(\delta/2)}\le\frac{4e\sqrt{m-1}}{\sqrt{\pi}}\left(\frac{\pi}{\sqrt{2}\delta}\right)^{m-1}\le\frac{1}{2}\left(\frac{\pi}{\delta}\right)^{m-1}.$$

Thus for $m \geq 12$ the cardinality of $\mathcal{N} \cup -\mathcal{N}$ is less than or equal to M.

Set

$$K = \operatorname{conv}\left\{ \left\{ \sum_I z_i \right\}_{|I|=k, I\subset\{1,\ldots,M\}} \right\}.$$

To continue the proof we need the following claim, which will be proved below.

Claim 3.5 *The body K, defined above, satisfies*

$$(k/2)B_2^m \subset K.$$

Now, let $s_0 = n/m$. Without loss of generality we can assume that s_0 is an integer. For every $1 \leq s \leq s_0$ define the operator $i_s : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ by $i_s e_j = e_l$, where $l = (s-1)m + j$. Then, taking $a_i^s = i_s z_i$, we have $a_i^s \in S^{n-1}$. Set $\{a_j\} = \{a_i^s\}_{i,s}$. By the choice of M and s_0 one has that the cardinality of the set $\{a_j\}$ is $Ms_0 \leq 2N$. Also, by construction, we have that $\{-a_i\} = \{a_i\}$, i.e. we need only half of a_i's to define B_k.

Denote $B = \operatorname{conv}\{i_s K\}_s$. Using the claim we obtain

$$B \supset (k/2) \sum_{s=1}^{s_0} \oplus B_2^m \supset \frac{k}{2\sqrt{s_0}} B_2^{s_0 m},$$

where $\sum \oplus$ denotes the ℓ_1-sum. It follows

$$B^0 \subset \frac{2\sqrt{s_0}}{k} B_2^n \subset \frac{2\sqrt{n}}{k\sqrt{m}} B_2^n.$$

It is not difficult to see that $B_k^0 \supset B$. Thus $B_k \subset 2\sqrt{n}/(k\sqrt{m})B_2^n$. Thus for every l-dimensional subspace E one has

$$|E \cap B_k| \leq \left| \frac{2\sqrt{n}}{k\sqrt{m}} B_2^l \right| \leq \left(C \frac{\sqrt{n}}{k\sqrt{ml}} \right)^l.$$

$\square$

Proof of the claim: Consider $u \in S^{m-1}$. Choose the minimal angle θ such that there are at least k points of the z_i's with $\langle u, z_i \rangle \geq \cos\theta$. Since $k \leq M/2$ we have $\theta \in [0, \pi/2]$. Assume that $\bar{z}_1, \bar{z}_2, \ldots$ are those points. Then we have

$$\left| \sum_{i=1}^k \bar{z}_i \right|^2 = \sum_{i,j=1}^k \langle \bar{z}_i, \bar{z}_j \rangle \geq k^2 \cos(2\theta) = k^2(1 - 2\sin^2\theta).$$

Denote $A = \{i \mid \langle \bar{z}_i, u \rangle > \cos\theta\}$. By minimality of θ we have $|A| \leq k$. Since $\{z_i\}_{i \leq M}$ is a δ-net in S^{m-1}, $\{\bar{z}_i\}_{i\in A}$ is a δ-net in $S(u, \theta - \delta)$. That implies

$$\sum_{i\in A} \nu_{m-1}(S(z_i, \delta)) \geq \nu_{m-1}(S(u, \theta - \delta)).$$

Using Fact 3.4 we obtain

$$k\delta \sin^{m-2}\delta \geq \frac{(\theta-\delta)\sin^{m-2}(\theta-\delta)}{e(m-1)}.$$

Thus $\sin^{m-1}(\theta-\delta) \leq e(m-1)k\delta^{m-1}$. Now if $\delta \geq \theta/2$ then $\theta \leq 2\delta \leq \pi/6$ and $1-2\sin^2\theta \geq 1/2$. If $\delta \leq \theta/2$ then

$$\sin\theta \leq 2\sin(\theta/2) \leq 2\sin(\theta-\delta) \leq 2(emk)^{1/(m-1)}\delta = 2\pi(emk/M)^{1/(m-1)}.$$

By the choice of m, M we have

$$\sin\theta \leq 2\pi\left(\frac{enk}{2N}\right)^{1/(m-1)} \leq 2\pi\left(\frac{e}{2\ 20^m}\right)^{1/(m-1)} \leq \frac{1}{2}.$$

Thus $1-2\sin^2\theta \geq 1/2$. That means that for every $u \in S^{m-1}$ there is $z = \sum_{i=1}^k \bar{z}_i$ such that $|z|^2 \geq k^2/2$ and $\langle u, z\rangle/|z| \geq \cos\theta \geq \sqrt{3}/2$.

The estimate now follows by the standard technique. Indeed, let b the best possible constant such that for every $x \in \mathbb{R}^n$ we have $\|x\|_K \leq b|x|$. Then for every $u \in S^{n-1}$ one has

$$\|u\|_K \leq \|z/|z|\,\|_K + \|u - z/|z|\,\|_K \leq$$

$$1/|z| + b\,|u - z/|z|\,| \leq \sqrt{2}/k + b(2-\sqrt{3}).$$

By minimality of b we obtain $b \leq \sqrt{2}/k + b(2-\sqrt{3})$, which means

$$b \leq \frac{\sqrt{2}}{(\sqrt{3}-1)k} \leq 2/k.$$

That proves the claim. □

4 Dvoretzky's theorem

First we recall the following version of Dvoretzky's Theorem (see Theorem 2.5 and Corollary 2.6 of [Go1]).

Theorem 4.1 *Let X be an n-dimensional space. Let $m \leq n$ and $G : \ell_2^m \longrightarrow X$ be the Gaussian operator defined by (1). Then*

$$\mathbf{E}\,\|G\| = \mathbf{E}\max_{|x|=1}\|Gx\| \leq E(X) + \sqrt{m}\varepsilon_2(X),$$

and

$$\mathbf{E}\min_{|x|\leq 1}\|Gx\| \geq E(X) - \sqrt{m}\varepsilon_2(X),$$

where $\varepsilon_2(X)$ is defined by (3).

In particular, if $E(X) > \sqrt{m}\varepsilon_2(X)$, then there exists an m-dimensional subspace $Y \subset X$ such that

$$d_Y \leq \mathbf{E}\,\|G\| \Big/ \mathbf{E} \min_{|x|\leq 1} \|Gx\| \leq \frac{E(X)+\sqrt{m}\varepsilon_2(X)}{E(X)-\sqrt{m}\varepsilon_2(X)}$$

Moreover, the subspace can be taken in a "random" way.

Theorem 4.2 *Let $\alpha \in (0,1)$ and $1 \leq m \leq n$. Let $\{a_j\}_{j\leq N} \subset S^{n-1}$ be such that $|a_i - a_j| \geq \varepsilon$ for every $i \neq j$ and let $X = (I\!R^n, B_k)$, $k \leq N$. There are absolute positive constants c and C such that*

(i) if

$$m \leq \alpha^2 \frac{c^2\varepsilon^2 k^2 \ln(3N/k)}{(\varepsilon_2(B_k))^2}$$

then there exists an m-dimensional subspace $Y \subset X$ satisfying

$$d_Y \leq \frac{1+\alpha}{1-\alpha};$$

(ii) if $\sqrt{m}\varepsilon_2(B_l) < c\varepsilon l\sqrt{\ln(3N/l)}$ then there exists an m-dimensional subspace $Y \subset X$ satisfying

$$d_Y \leq \max\{1,\ l/k\}\,\frac{Ck\sqrt{\ln(3N/k)}+\sqrt{m}\varepsilon_2(B_k)}{c\varepsilon l\sqrt{\ln(3N/l)}-\sqrt{m}\varepsilon_2(B_l))}.$$

Moreover, the subspaces can be taken in a "random" way.

Proof: The first part of the theorem follows by Theorem 4.1 and Lemma 2.3.

To show the second part of the theorem note that $\|x\|_s \leq \|x\|_k \leq (k/s)\|x\|_s$ for every $s \leq k$. Let G be the Gaussian operator $G : \ell_2^m \longrightarrow X$. Then by Theorem 4.1 and Lemma 2.3

$$\mathbf{E}\,\|G\| \leq Ck\sqrt{\ln(3N/k)} + \sqrt{m}\varepsilon_2(B_k).$$

If $l \geq k$ then

$$\mathbf{E} \min_{|x|=1} \|Gx\|_k \geq \max_{l\geq k} \frac{k}{l}\ \mathbf{E} \min_{|x|=1} \|Gx\|_l \geq$$

$$\max_{l\geq k} \frac{k}{l}\,(c\varepsilon l\sqrt{\ln(3N/l)} - \sqrt{m}\varepsilon_2(B_l)).$$

If $l \leq k$ then

$$\mathbf{E} \min_{|x|=1} \|Gx\|_k \geq \max_{l\leq k}\ \mathbf{E} \min_{|x|=1} \|Gx\|_l \geq$$

$$\max_{l\le k}(c\varepsilon l\sqrt{\ln(3N/l)}-\sqrt{m}\varepsilon_2(B_l)).$$

The result follows by Theorem 4.1. $\square$

Remark 1. The general case (when $\{a_i\}\not\subset S^{n-1}$) can be treated using Corollary 2.5.

Remark 2. As $\varepsilon_2(B_k)=\max_{|x|=1}\sum|\langle x,a_i\rangle|^*=\max_{\{\pm,|I|=k\}}|\sum_{i\in I}\pm a_i|$ and likewise for $\varepsilon_2(B_l)$, we may replace $\varepsilon_2(B_k)$ and $\varepsilon_2(B_l)$ by these values.

5 Properties of the spaces intermediate between ℓ_1 and ℓ_∞

In this section we investigate the spaces whose unit balls are convex hulls of $\{B_1^n,(B_\infty^n/k)\}$ and their duals, $(kB_1^n)\cap B_\infty^n$, where B_1^n denotes the unit ball of l_1^n and B_∞^n the unit ball of l_∞^n. As we shall see in Lemma 5.1 those spaces are particular cases of spaces with unit balls B_k and B_k^0, when $N=n$ and the sequence $\{a_i\}_i$ is $\{e_i\}_i$. Henceforth B_k will refer to this choice of the sequence $\{a_i\}_i$.

In the first subsection we investigate Dvoretzky's theorem, type and cotype constants of such spaces. In the second subsection we provide asymptotically sharp estimates of the volume ratio of B_k and B_k^0, and of the projection constant of B_k. Finally, in the third section we investigate the p-summing norm of the identity operator acting on some special spaces. As a corollary we obtain asymptotically sharp estimates of the projection constant of B_k^0.

Lemma 5.1 *Let $\{a_i\}_{i\le n}=\{e_i\}_{i\le n}$ and $k\le n$. Then*

$$B_k=\mathrm{conv}\ \{B_1^n,(B_\infty^n/k)\}\quad\textit{and}\quad B_k^0=(kB_1^n)\cap B_\infty^n.$$

Proof: Denote $B:=(kB_1^n)\cap B_\infty^n$. Fix x. Without loss of generality assume that only k terms of $\{|x_i|\}$ are larger than or equal to x_k^*. Define $z=\{z_i\}$ by

$$z_i=\begin{cases}\text{sign } x_i & \text{for } |x_i|\ge x_k^*,\\ 0 & \text{otherwise.}\end{cases}$$

Clearly $z\in B$, and hence

$$\|x\|_{B^0}=\max_{y\in B}\langle x,y\rangle\ge\langle x,z\rangle=\sum_{i=1}^k|x_i|^*=\|x\|_k.$$

To get the inequality in the other direction, assume, as we can, that $x_1\ge x_2\ge\ldots\ge x_n\ge 0$. Then for every $y\in B$ one has $\langle x,y\rangle\le\sum_1^k x_i^*$, i.e.

$\|x\|_{B^0} \leq \|x\|_k$. That proves the first equality. The second follows by duality. $\square$

Below we will use Khinchine's inequality, which states that there exists an absolute constant $c > 0$ such that for every $p \geq 1$ and every $\{b_i\}_i \subset I\!R$ one has

$$\frac{1}{A_p}\left(\frac{1}{2^n}\sum_{\varepsilon}\left|\sum_{i=1}^{n} b_i\varepsilon_i\right|^p\right)^{1/p} \leq \left(\frac{1}{2^n}\sum_{\varepsilon}\left|\sum_{i=1}^{n} b_i\varepsilon_i\right|^2\right)^{1/2} = \tag{8}$$

$$\left(\sum_{i=1}^{n}|b_i|^2\right)^{1/2} \leq B_p\left(\frac{1}{2^n}\sum_{\varepsilon}\left|\sum_{i=1}^{n} b_i\varepsilon_i\right|^p\right)^{1/p},$$

where the sum is taken over all $\varepsilon \in \{-1,1\}^n$ and

$$A_p \leq \begin{cases} 1 & \text{for } 1 \leq p \leq 2, \\ c\sqrt{p} & \text{for } p > 2, \end{cases}$$

$$B_p = \begin{cases} \sqrt{2} & \text{for } 1 \leq p \leq 2, \\ 1 & \text{for } p > 2. \end{cases}$$

5.1 Consequences of Dvoretzky's theorem. Type 2 and Cotype 2

We start with a well known estimate of type and cotype constants (see e.g. [Pi1], [T]). Let $C_2(B)$ and $T_2(B)$ denote the cotype and type constant of the space with the unit ball B.

Lemma 5.2 *Let $B \subset I\!R^n$ be a convex body with $m > 1$ extreme points. Then*

$$C_2\left(B^0\right) \leq T_2\left(B\right) \leq c\sqrt{\ln m},$$

where c is an absolute constant.

In the following statements we describe the properties of B_k and B_k^0. We start with a trivial fact.

Fact 5.3 *The following sharp inclusions hold*

$$\min\left\{1, k/\sqrt{n}\right\} B_2^n \subset B_k^0 \subset \sqrt{k} B_2^n$$

and

$$\frac{1}{\sqrt{k}} B_2^n \subset B_k \subset \max\left\{1, \sqrt{n}/k\right\} B_2^n.$$

In particular it means $\varepsilon_2(B_k) = \sqrt{k}$ and $\varepsilon_2(B_k^0) = \max\{1, \sqrt{n}/k\}$.

We shall also use the following estimates.

Lemma 5.4 *There are absolute positive constants c and C such that*

$$c\ k\ \sqrt{\ln(2n/k)} \le E(B_k) \le C\ k\ \sqrt{\ln(2n/k)}$$

and

$$c\ \left(\sqrt{\ln n} + n/k\right) \le E\left(B_k^0\right) \le C\ \left(\sqrt{\ln n} + n/k\right).$$

Proof: The first estimate is a consequence of Lemma 2.1 (and the remark after it). The second estimate can be obtained directly, since

$$1/2\ (|x|_\infty + |x|_1/k) \le \|x\|_{B_k^0} = \max\{|x|_\infty, |x|_1/k\} \le |x|_\infty + |x|_1/k,$$

where $|\cdot|_1$ is the l_1-norm and $|\cdot|_\infty$ is the l_∞-norm. □

The next two corollaries give Dvoretzky type theorems. They follow immediately from Theorem 4.2.

Corollary 5.5 *Let $1 \le m \le n$ and $1 \le k \le n$. Let $\delta \in (0,1)$. There are absolute constants c and C such that*

(i) if $m \le ck\delta^2 \ln(3n/k)$ then there exists an m-dimensional subspace $E \subset I\!R^n$ such that

$$d\left(B_k \cap E, B_2^m\right) \le \frac{1+\delta}{1-\delta};$$

(ii) if $m \ge k \ln(3n/k)$ then there exists an m-dimensional subspace $E \subset I\!R^n$ such that

$$d\left(B_k \cap E, B_2^m\right) \le C\sqrt{\frac{m}{k\ \ln(3n/m)}}.$$

Moreover, the subspaces can be taken in a "random" way.

Proof: By the definition of the body B_k we have $|a_i - a_j| = \sqrt{2}$ and by Fact 5.3 $\varepsilon_2(B_k) = \sqrt{k}$. Thus the first estimate follows by Theorem 4.2.

If $m \ge k\ \ln(3n/k)$ then $k \le \frac{m}{\ln(3n/k)}$. On the other hand there exists an absolute constant $c_1 \ge 1$ such that if $l = \frac{c_1 m}{\ln(3n/m)}$ then $c_0\sqrt{2}\sqrt{\ln(3n/l)}l \ge 2\sqrt{m}\sqrt{l}$. Since $l \ge k$, $\varepsilon_2(B_k) = \sqrt{l}$ and $|a_i - a_j| = \sqrt{2}$ for $i \ne j$, we can apply the "Moreover" part of Theorem 4.2 in order to obtain an m-dimensional subspace $Y \subset X$ with

$$d_Y \le \frac{ck\sqrt{\ln(3n/k)} + \sqrt{mk}}{ck\sqrt{2}\sqrt{\ln(3n/l)}} \le C_1 \frac{\sqrt{m}}{\sqrt{k}\sqrt{\ln(3n/(c_1 m))}},$$

which implies the desired estimate. □

Corollary 5.6 *Let $1 \le m \le n$ and $1 \le k \le n$. There are an absolute constant C and an m-dimensional subspace $E \subset I\!R^n$ such that*

(i) if $k \le n/\sqrt{\ln(2n/m)}$ then

$$d\left(B_k^0 \cap E, B_2^m\right) \le C\left(1 + k\ \sqrt{m}/n\right);$$

(ii) if $k > n/\sqrt{\ln(2n/m)}$ then

$$d\left(B_k^0 \cap E, B_2^m\right) \le C\ \left(1\ +\ \sqrt{\frac{m}{\ln\left(2n/m\right)}}\right).$$

Moreover, the subspaces can be taken in a "random" way.

Remark. The second estimate is known in a general case. That is for every convex body K and every $m \le n/2$ there exists an m-dimensional subspace E, such that

$$d\left(K \cap E, B_2^m\right) \le C\ \left(1\ +\ \sqrt{\frac{m}{\ln\left(2n/m\right)}}\right)$$

([MS3], see also [MS2, GGM, LiT, Gu]).

Proof:
Note that for every $1 \le l \le n$

$$\|x\|_{B_k^0} = \max\ \left\{\frac{|x|_1}{k}, |x|_\infty\right\} \ge \frac{|x|_1}{2k} + \frac{|x|_\infty}{2} \ge \frac{1}{2}\ \left(\frac{|x|_1}{k} + \frac{1}{l}\ \ \sum_{i\le l}|x_i|^*\right),$$

where $|\cdot|_1$ is the l_1-norm and $|\cdot|_\infty$ is the l_∞-norm. Using the estimates for the Gaussian operator $G : l_2^m \to (I\!R^n, B_k^0)$, Fact 5.3 and Lemma 5.4, we get

$$\mathbf{E}\,\|G\| \le E\left(B_k^0\right) + \sqrt{m}\varepsilon_2\left(B_k^0\right) \le \begin{cases} c\sqrt{\ln n} + \sqrt{m} & \text{for } k > n/\sqrt{\ln(2n)} \\ c\frac{n}{k} + \sqrt{m} & \text{for } \sqrt{n} \le k \le n/\sqrt{\ln(2n)} \\ c\frac{n}{k} & \text{for } k < \sqrt{n}. \end{cases}$$

Consider now the new norm defined by

$$|||x||| = \frac{|x|_1}{k} + \frac{1}{l}\sum_{i=1}^{l}|x_i|^* \le 2\ \|x\|_{B_k^0}.$$

Clearly $\varepsilon_2\left((I\!R^n, |||\cdot|||)\right) \le \frac{\sqrt{n}}{k} + \frac{1}{\sqrt{l}}$. Therefore, by Theorem 4.1 we have

$$\mathbf{E}\inf_{|x|=1} 2\ \|Gx\|_{B_k^0} \ge \mathbf{E}\inf_{|x|=1}|||Gx||| \ge \mathbf{E}\left|\left|\left|\sum_{i\le n} g_i\ e_i\right|\right|\right| - \sqrt{m}\varepsilon_2\left((I\!R^n, |||\cdot|||)\right) \ge$$

$$\frac{n}{k}\sqrt{\frac{2}{\pi}} + c\sqrt{\ln(2n/l)} - \sqrt{m}\left(\frac{\sqrt{n}}{k} + \frac{1}{\sqrt{l}}\right) =$$
$$\frac{\sqrt{n}}{k}\left(\sqrt{2n/\pi} - \sqrt{m}\right) + \frac{c\sqrt{l}\sqrt{\ln(2n/l)} - \sqrt{m}}{\sqrt{l}}.$$

We can assume that $m \leq n/2$ (otherwise the corollary is obvious). Choose l satisfying $c\sqrt{l}\sqrt{\ln(2n/l)} \approx \sqrt{2m}$, so that $l \approx \frac{m}{\ln(2n/m)}$. Then

$$\mathbf{E} \inf_{|x|=1} \|Gx\|_{B_k^0} \geq c_1 \left(n/k + \sqrt{\ln(2n/m)}\right)$$

for some absolute constant $c_1 > 0$. Thus for $k \leq n/\sqrt{\ln(2n/m)}$ one has

$$d_E \leq \frac{\mathbf{E}\,\|G\|}{\mathbf{E}\inf_{|x|=1}\|Gx\|_{B_k^0}} \leq \frac{c\; n/k + \sqrt{m}}{2c_1 n/k} \leq C\left(1 + \frac{k\sqrt{m}}{n}\right).$$

That proves the first case. If $k \geq n/\sqrt{\ln(2n/m)}$ one has

$$d_E \leq \frac{\mathbf{E}\,\|G\|}{\mathbf{E}\inf_{|x|=1}\|Gx\|_{B_k^0}} \leq \frac{\sqrt{m} + c\sqrt{\ln m}}{2\sqrt{\ln(2n/m)}} \leq C\left(1 + \sqrt{\frac{m}{\ln(2n/m)}}\right),$$

which proves the second case. □

For the proof Proposition 5.8 as well as in Section 7 we need the following result of S. Kwapień and C. Schütt (Corollary 2.3 of [KS2], see also Theorem 1.2 of [KS1]).

Lemma 5.7 *(i) Let $q \geq 1$. Let $b \in I\!R^n$ such that $b_1 \geq b_2 \geq \cdots \geq b_n \geq 0$. For every $x \in I\!R^n$ one has*

$$\frac{1}{5n}\sum_{j=1}^{n} s(j) + \left(\frac{1}{n}\sum_{j=n+1}^{n^2} s(j)^q\right)^{1/q} \leq \frac{1}{n!}\sum_{\pi}\left(\sum_{i=1}^{n} |x_i b_{\pi(i)}|^q\right)^{1/q} \leq$$
$$\frac{1}{n}\sum_{j=1}^{n} s(j) + \left(\frac{1}{n}\sum_{j=n+1}^{n^2} s(j)^q\right)^{1/q},$$

where $\{s(k)\}_k$ is the non-increasing rearrangement of $\{|x_i b_j|\}_{i,j}$.

(ii) Let $q \geq 1$. Let $b \in I\!R^n$ such that $b_1 \geq b_2 \geq \cdots \geq b_n > 0$ and such that $\sum_{1\leq i\leq n} b_i = n$. For every $x \in I\!R^n$ one has

$$\frac{1}{4}\left(\frac{1}{2} - \frac{1}{n-1}\right)\|x\|_{N_b} \leq \frac{1}{n!}\sum_{\pi}\left(\sum_{i=1}^{n} |x_i b_{\pi(i)}|^q\right)^{1/q} \leq 8\left(1 + \frac{2}{n-1}\right)\|x\|_{N_b},$$

where N_b is the Orlicz function defined by

$$\frac{1}{5}N_b^{*-1}\left(\frac{l}{n}\right) \leq \frac{1}{n}\left\{\sum_{1\leq i\leq l} b_i + l^{\frac{q-1}{q}}\left(\sum_{l+1\leq i\leq n} b_i^q\right)^{1/q}\right\} \leq 2N_b^{*-1}\left(\frac{l+1}{n}\right).$$

The following Proposition provides estimates for the type and cotype constants of the bodies.

Proposition 5.8 *There are absolute constants c and C such that*

(i) for $k < \sqrt{n}$ we have

$$c\sqrt{k} \leq C_2\left(B_k^0\right) \leq T_2\left(B_k\right) \leq \min\left\{\sqrt{n/k},\ C\sqrt{k}\sqrt{\ln n}\right\}$$

and

$$c\sqrt{n/k} \leq T_2\left(B_k^0\right) \leq \sqrt{n/k};$$

(ii) for $k \geq \sqrt{n}$ we have

$$c\sqrt{k} \leq C_2\left(B_k^0\right) \leq T_2\left(B_k\right) \leq \sqrt{k}$$

and

$$c\ \max\left\{\sqrt{\ln\ k},\ \sqrt{n/k}\right\} \leq T_2\left(B_k^0\right) \leq \min\left\{\sqrt{k}, \sqrt{n/k}\ \sqrt{\ln n}\right\};$$

(iii) for all k we have

$$c\sqrt{n/k} \leq C_2\left(B_k\right) \leq C\sqrt{n/k}.$$

Proof:

(i) To prove the first upper estimate note that the set of extreme points of B_k^0 is the set of points $x = \{x_i\}$ satisfying

$$x_i = \begin{cases} \pm 1 & \text{for } i \in A \\ 0 & \text{otherwise,} \end{cases}$$

for some set A with cardinality $|A| = k$. Therefore by Lemma 5.2 we have

$$C_2\left(B_k^0\right) \leq T_2\left(B_k\right) \leq c\sqrt{\ln\left(2^k\binom{n}{k}\right)} \leq c_1\sqrt{k}\sqrt{\ln(2n/k)},$$

and as $k < \sqrt{n}$ we have that $\ln(2n/k)$ is, up to a numerical constant, of the same order as $\ln n$.

Since $T_2(B_k) \le d_{B_k}$, we get with Fact 5.3 in the case $k < \sqrt{n}$ that

$$T_2(B_k) \le \sqrt{n/k}.$$

This is the other upper estimate.

To obtain the lower estimate for $C_2(B_k^0)$ it is enough to take the set of points $x_i = e_i$, $i \le k$. This works for all $1 \le k \le n$.

Again, since $T_2(B_k^0) \le d_{B_k}$, we get with Fact 5.3 in the case $k < \sqrt{n}$ that

$$T_2\left(B_k^0\right) \le \sqrt{n/k}.$$

The lower estimate for $T_2(B_k^0)$ will follow from (iii), since $T_2(B_k^0) \ge C_2(B_k)$.

(ii) We have that $C_2(B_k^0) \le T_2(B_k)$ and the upper estimate for $T_2(B_k)$ follows again from Fact 5.3 as now $k \ge \sqrt{n}$. The lower estimate for $C_2(B_k^0)$ is as in (i).

The first upper estimate $T_2(B_k^0) \le \sqrt{k}$ follows again from the fact that $T_2(B_k^0) \le d_{B_k}$ and Fact 5.3.

The other upper estimate for $T_2(B_k^0)$ follows from (iii) and the fact that $T_2(B_k^0) \le \sqrt{\ln n}\, C_2(B_k)$ (see [Pi2]).

Taking the k-dimensional subspace $E = \mathrm{span}\{e_i\}_{i \le k}$ of $I\!R^n$ one can easily check that $l_\infty^k \subset (I\!R^n, B_k)$. Therefore

$$T_2\left(B_k^0\right) \ge \sqrt{\ln\ k},$$

which gives the first lower estimate for $T_2(B_k^0)$. As $T_2(B_k^0) \ge C_2(B_k)$, the other lower estimate for $T_2(B_k^0)$ for all k will follow from (iii).

(iii) We get the lower estimate for $C_2(B_k)$ by taking $\left[\frac{n}{k}\right]$ vectors x_i of the following form

$$x_1 = (1, \ldots, 1, 0, \ldots, 0),$$

with 1 on the first k coordinates, 0 on the others;

$$x_2 = (0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0),$$

with 0 on the first k coordinates, 1 on the next k and 0 on the others. We continue with the other x_i in the obvious way.

To prove the upper bound for $C_2(B_k)$ we consider for $x \in I\!R^n$ the norm

$$|||x||| = \sum_{i \le k} |x_i|^* \quad + \sqrt{k} \left(\sum_{k+1 \le i \le n} |x_i|^{*2} \right)^{\frac{1}{2}}.$$

Since $|x_{k+1}|^* \le \frac{1}{k}\|x\|_k$, we obtain

$$\|x\|_k \le |||x||| \le \|x\|_k + \sqrt{k}\sqrt{n-k}|x_{k+1}|^* \le$$

$$\|x\|_k \left(1+\sqrt{\frac{n-k}{k}}\right) \le 2\sqrt{n/k}\|x\|_k.$$

Thus the Banach-Mazur distance is

$$d\left((\mathbb{R}^n, |||\cdot|||), (\mathbb{R}^n, \|\cdot\|_k)\right) \le 2\sqrt{\frac{n}{k}}.$$

Now we show that $(\mathbb{R}^n, |||\cdot|||)$ has cotype 2. This will follow once we have shown that $(\mathbb{R}^n, |||\cdot|||)$ is c-isomorphic to a subspace of L^1 where c does not depend on the dimension n and on k. This is what we shall prove now: By Lemma 5.7 we have for all $n \in \mathbb{N}$, $b, x \in \mathbb{R}^n$

$$\frac{1}{5n}\sum_{j=1}^{n} s(j) + \left(\frac{1}{n}\sum_{j=n+1}^{n^2} s(j)^2\right)^{1/2} \le \frac{1}{n!}\sum_{\pi}\left(\sum_{i=1}^{n}|x_i b_{\pi(i)}|^2\right)^{1/2} \le$$

$$\frac{1}{n}\sum_{j=1}^{n} s(j) + \left(\frac{1}{n}\sum_{j=n+1}^{n^2} s(j)^2\right)^{1/2},$$

where $\{s(k)\}_k$ is the non-increasing rearrangement of $\{|x_i b_j|\}_{i,j}$.

Using Khinchine's inequality (8) one can prove that $(\mathbb{R}^n, \|\cdot\|)$ is equivalent to a subspace of L^1, where

$$\|x\| = \frac{1}{n!}\sum_{\pi}\left(\sum_{i=1}^{n}|x_i b_{\pi(i)}|^2\right)^{1/2}.$$

We choose $b = (1, \ldots, 1, 0, \ldots, 0)$ with m coordinates equal to 1 and the others equal to 0. Assume that n/m is an integer. Then

$$\sum_{j=1}^{n} s(j) = m\sum_{j=1}^{n/m}|x_i|^* \quad \text{and} \quad \sum_{j=n+1}^{n^2} s(j)^2 = m\sum_{j=(n/m)+1}^{n}|x_i|^{*2}.$$

Hence

$$\frac{1}{n}\sum_{j=1}^{n} s(j) + \left(\frac{1}{n}\sum_{j=n+1}^{n^2} s(j)^2\right)^{1/2} = \frac{m}{n}\sum_{j=1}^{n/m}|x_i|^* + \sqrt{\frac{m}{n}}\left(\sum_{j=(n/m)+1}^{n}|x_i|^{*2}\right)^{\frac{1}{2}}.$$

Without loss of generality we may assume that n/k is an integer. Choosing m such that $k = \frac{n}{m}$ we obtain that $(\mathbb{R}^n, |||\cdot|||)$ is isomorphic to a subspace of L^1.

This proves the proposition. $\square$

5.2 Volume Ratios. Projection constants of B_k^0

We start with the estimate of the volume ratio of the bodies. Let us first recall that for a body $K \subset I\!R^n$ the volume ratio $\mathrm{vr}(K)$ is

$$\mathrm{vr}(K) = \left(|K| \,/\, |\mathcal{E}|\right)^{1/n},$$

where $\mathcal{E}$ is the ellipsoid of maximal volume in K.

By the volume ratio of the space we mean the volume ratio of its unit ball.

Lemma 5.9 *There exist absolute constants $c > 0$ and C such that*

$$c\sqrt{n/k} \le \mathrm{vr}\,(B_k) \le C\sqrt{n/k}$$

and

$$c \max\left\{\frac{k}{\sqrt{n}}, 1\right\} \le \mathrm{vr}\left(B_k^0\right) \le C \max\left\{\frac{k}{\sqrt{n}}, 1\right\}.$$

Proof: Since the bodies have enough symmetries, the ellipsoids in Fact 5.3 are of maximal volume. To estimate the volume of B_k note that $B_\infty^n/k \subset B_k \subset (n/k)\, B_1^n$. Hence

$$(2/k)^n \le |B_k| \le (2e/k)^n$$

and, by Santaló inequality and inverse Santaló inequality [BM],

$$(c_1 k/n)^n \le \left|B_k^0\right| \le (c_2 k/n)^n .$$

This implies the result. $\square$

Remark. This result should be compared with the corollary of Theorem 7 of [GGMP] which says

$$\mathbf{E}\ \mathrm{vr}\,(K \cap E) \ge \frac{c\,\sqrt{l}}{E(K)\ \max_i \|e_i\|_{K^0}},$$

where the expectation $\mathbf{E}$ is taken with respect to the normalized Haar measure on the Grassmanian of all l-dimensional subspaces $E \subset I\!R^n$. By Lemma 5.4 we obtain

$$\mathbf{E}\ \mathrm{vr}\,(B_k \cap E) \ge \frac{c\,\sqrt{l}}{k\sqrt{\ln(3n/k)}}$$

and

$$\mathbf{E}\ \mathrm{vr}\left(B_k^0 \cap E\right) \ge \frac{c\,\sqrt{l}}{\sqrt{\ln n} + n/k}.$$

The volume ratio estimates allow us to obtain the following estimates for the projection constant λ. Recall that for every n-dimensional body K one has $\lambda(K) \le \sqrt{n}$ and $\lambda(K) \le d(K, B_\infty^n)$.

Theorem 5.10 *There is an absolute constant $c > 0$ such that*

(i) for every $\sqrt{n} \leq k \leq n$ one has

$$c\frac{n}{k} \leq \lambda\left(B_k^0\right) \leq \frac{n}{k},$$

(ii) for every $1 \leq k \leq \sqrt{n}$ one has

$$c\sqrt{n} \leq \lambda\left(B_k^0\right) \leq \sqrt{n}.$$

Proof: To prove the first upper estimate we use a well-known estimate $\lambda(K) \leq d(K, B_\infty^n)$. The estimate $d\left(B_k^0, B_\infty\right) \leq \frac{n}{k}$ is trivial. Since $\lambda(K) \leq \sqrt{n}$ for every $K \subset \mathbb{R}^n$ we obtain the second upper estimate.

To obtain the lower estimates we use the following inequalities from [GMP]. For every n-dimensional normed space X one has

$$\sqrt{n} \leq evr\left(X, \ell_\infty\right) \mathrm{vr}\left(X\right) \leq \sqrt{en}$$

and

$$evr\left(X, \ell_\infty\right)\ zr\left(X\right) \leq \lambda\left(X\right),$$

where evr and zr denote the external volume ratio and zonoid ratio correspondingly (see e.g. [GMP] for the precise definitions). For every n-dimensional normed space X with a 1-unconditional basis one has

$$1 \leq zr\left(X\right) zr\left(X^*\right) \leq C,$$

where C is a numerical constant. Since $X = (\mathbb{R}^n, B_k)$ and $X^* = (\mathbb{R}^n, B_k^0)$ have a 1-unconditional basis we obtain

$$\lambda\left(B_k^0\right) \geq c\frac{\sqrt{n}}{\mathrm{vr}\left(B_k^0\right)}.$$

The result follows by Lemma 5.9. □

Remark. The estimate on $\mathrm{vr}\left(B_k^0\right)$ allows us to obtain lower bounds for the GL-constant gl_2 of subspaces of B_k^0 since by [GJ2], for every n-dimensional normed space X there exists a subspace $Y \subset X$, dim $Y \leq \frac{n}{2}$, such that $\mathrm{vr}\left(X\right) \leq c\ zr\left(Y\right)$ and $zr\left(Y\right) zr\left(Y^*\right) \leq c_1 \mathrm{gl}_2\left(Y\right)$, where c and c_1 are positive absolute constants ([GMP]).

5.3 p-summing norms and related invariants

We shall obtain the estimates for the projection constant of B_k as a corollary of the following lemma, in which we compute the p-summing norm $\pi_p(K)$ of the identity operator $Id : (\mathbb{R}^n, K) \longrightarrow (\mathbb{R}^n, K)$ for some special bodies K. Recall that $\pi_p(K)$ is the best possible constant, satisfying

$$\sum_{j=1}^{N} \|y_j\|_K^p \leq \pi_p^p(K) \sup_{\|f\|_* \leq 1} \sum_{j=1}^{N} |\langle y_j, f\rangle|^p$$

for every N and every $y_1, y_2, ..., y_N \in E = (\mathbb{R}^n, K)$, where $\|\cdot\|_*$ denotes the norm in $E^* = (\mathbb{R}^n, K^0)$.

Let $b \in \mathbb{R}^n$ be such that $b_1 \geq b_2 \geq ... \geq b_n \geq 0$ and $b_1 > 0$. Define the norm $\|x\|_b = \sum_i b_i |x_i|^*$.

Lemma 5.11 *Let K_b be the unit ball of $\|\cdot\|_b$. Let $p \geq 1$. Then there is an absolute constant c such that*

$$\frac{1}{A_p^p} \sup_{x\neq 0} \frac{n!\ \|x\|_b^p}{\sum_\pi \left(\sum_{i=1}^n |x_i b_{\pi(i)}|^2\right)^{p/2}} \leq \pi_p^p(K_b) \leq B_p^p \sup_{x\neq 0} \frac{n!\ \|x\|_b^p}{\sum_\pi \left(\sum_{i=1}^n |x_i b_{\pi(i)}|^2\right)^{p/2}},$$

where

$$A_p \leq \begin{cases} 1 & \text{for } 1 \leq p \leq 2, \\ c\sqrt{p} & \text{for } p > 2, \end{cases}$$

$$B_p = \begin{cases} \sqrt{2} & \text{for } 1 \leq p \leq 2, \\ 1 & \text{for } p > 2. \end{cases}$$

Proof: Given $x \in \mathbb{R}^n$, $\varepsilon \in \{-1,1\}^n$, and a permutation π of $\{1,2,...,n\}$ denote the vector $(\varepsilon_i x(\pi(i)))_{i=1}^n$ by εx^π.

We show first the left hand inequality. Let x be a vector for which the supremum

$$\sup_{x\neq 0} \frac{n!\ \|x\|_b^p}{\sum_\pi \left(\sum_{i=1}^n |x_i b_{\pi(i)}|^2\right)^{p/2}}$$

is attained. As a sequence we choose $\{\varepsilon x^\pi\}_{\varepsilon,\pi}$. Then we have

$$\sum_\pi \sum_\varepsilon \|\varepsilon x^\pi\|_b^p \leq \pi_p^p(K_b) \sup_{\|f\|_*=1} \sum_\pi \sum_\varepsilon | < \varepsilon x^\pi, f > |^p.$$

This means

$$\|x\|_b^p \leq \pi_p^p(K_b) \sup_{\|f\|_*=1} \frac{1}{n! 2^n} \sum_\pi \sum_\varepsilon \left| \sum_{i=1}^n \varepsilon_i x(\pi(i)) f(i) \right|^p.$$

We apply now Khintchine-inequality (8)

$$\|x\|_b^p \leq A_p^p \pi_p^p(K_b) \sup_{\|f\|_*=1} \frac{1}{n!} \sum_{\pi} \left(\sum_{i=1}^{n} |x(\pi(i)) f(i)|^2 \right)^{\frac{p}{2}}.$$

Instead of taking the supremum over all f with norm 1 we may take only the supremum over the extreme points of the unit ball of $\|\cdot\|_*$. The extreme points of the unit ball are all the points εb^π where ε ranges over all sequences of signs and π over all permutations. Thus we get

$$\|x\|_b^p \leq A_p^p \pi_p^p(K_b) \frac{1}{n!} \sum_{\pi} \left(\sum_{i=1}^{n} |x(\pi(i)) b_i|^2 \right)^{\frac{p}{2}}.$$

Now we proof the right hand inequality. Let $\delta > 0$ and let $y_i \in \mathbb{R}^n$, $1 \leq i \leq m$ be a sequence such that

$$\sum_{j=1}^{m} \|y_j\|_b^p \geq (\pi_p^p(K_b) + \delta) \sup_{\|f\|_*=1} \sum_{j=1}^{m} | < y_j, f > |^p.$$

Since the norm is 1-symmetric we get for all sequences of signs ε and permutations π

$$\sum_{j=1}^{m} \|\varepsilon y_j^\pi\|_b^p \geq (\pi_p^p(K_b) + \delta) \sup_{\|f\|_*=1} \sum_{j=1}^{m} | < \varepsilon y_j^\pi, f > |^p.$$

By triangle-inequality

$$\sum_{j=1}^{m} \|y_j\|_b^p \geq (\pi_p^p(K_b) + \delta) \sup_{\|f\|_*=1} \frac{1}{n! 2^n} \sum_{\pi} \sum_{\varepsilon} \sum_{j=1}^{m} | < \varepsilon y_j^\pi, f > |^p.$$

As in the proof of the left hand inequality we apply Khintchine-inequality (8)

$$\sum_{j=1}^{m} \|y_j\|_b^p \geq B_p^p (\pi_p^p(K_b) + \delta) \sup_{\|f\|_*=1} \frac{1}{n!} \sum_{\pi} \sum_{j=1}^{m} \left(\sum_{i=1}^{n} |y_j(\pi(i)) f(i)|^2 \right)^{\frac{p}{2}}.$$

Since $\|b\|_* = 1$

$$\sum_{j=1}^{m} \|y_j\|_b^p \geq B_p^p (\pi_p^p(K_b) + \delta) \frac{1}{n!} \sum_{\pi} \sum_{j=1}^{m} \left(\sum_{i=1}^{n} |y_j(\pi(i)) b_i|^2 \right)^{\frac{p}{2}}.$$

It follows that

$$B_p^p \pi_p^p(K_b) \leq \sup_{\{y_j\}_{j=1}^m} \frac{\sum_{j=1}^m \|y_j\|_b^p}{\sum_{j=1}^m \frac{1}{n!} \sum_\pi \left(\sum_{i=1}^n |y_j(\pi(i)) b_i|^2 \right)^{\frac{p}{2}}}$$

where the supremum is taken over all sequences of vectors such that at least one vector is different from 0. It is left to observe that the supremum is attained for a sequence consisting of one vector only.

$$B_p^p \pi_p^p(K_b) \le \sup_{y \neq 0} \frac{\|y\|_b^p}{\frac{1}{n!}\sum_\pi \left(\sum_{i=1}^n |y(\pi(i))b_i|^2\right)^{\frac{p}{2}}}$$

□

Proposition 5.12 *Let $1 \le p \le 2$. Let $b \in I\!R^n$ with $b_1 \ge b_2 \ge \dots b_n > 0$ and let E_b be $I\!R^n$ with the norm $\|x\|_b = \sum_i b_i |x_i|^*$. Then we have*

$$c_1 \ \pi_p(K_b) \le \sup_{x \neq 0} \frac{\|x\|_b}{\|(|x_1|^p, \dots, |x_n|^p)\|_{N_{b^p}}^{1/p}} \le c_2 \ \pi_p(K_b),$$

where c_1 and c_2 are positive absolute constants and where the Orlicz function N_{b^p} is defined by

$$\frac{1}{5} N_{b^p}^{*-1}\left(\frac{l}{n}\right) \le \frac{1}{n}\left\{\sum_{1 \le i \le l} b_i^p + l^{\frac{2-p}{2}}\left(\sum_{l+1 \le i \le n} b_i^2\right)^{\frac{p}{2}}\right\} \le 2 N_{b^p}^{*-1}\left(\frac{l+1}{n}\right).$$

In particular for $p = 1$ we get

$$c_1 \ \pi_1(K_b) \le \|b\|_{N_b^*} \le c_2 \ \pi_1(K_b),$$

where c_1 and c_2 are constants and where the Orlicz function N_b is defined by

$$\frac{1}{5} \ N_b^{*-1}\left(\frac{l}{n}\right) \ \le \ \frac{1}{n}\sum_{i \le l} b_i + \sqrt{l}\ \left(\sum_{l+1 \le i \le n} b_i^2\right)^{\frac{1}{2}} \ \le \ 2 \ N_b^{*-1}\left(\frac{l+1}{n}\right).$$

Proof: We may assume that $\sum_{1 \le i \le n} b_i^p = n$. By Lemma 5.11 we have

$$\frac{\pi_p(K_b)}{B_p} \le \ \sup_{x \neq 0} \frac{\|x\|_b}{\left[\frac{1}{n!}\sum_\pi \left(\sum_{i=1}^n |x_i b_{\pi(i)}|^2\right)^{p/2}\right]^{1/p}} \ \le A_p \ \pi_p(K_b).$$

By Lemma 5.7, applied for $q = 2/p$, we obtain

$$(1/4)^{1/p}\left(\frac{1}{2} - \frac{1}{n-1}\right)^{1/p} \|(|x_1|^p, \dots, |x_n|^p)\|_{N_{b^p}}^{1/p}$$

$$\le \left[\frac{1}{n!}\sum_\pi \left(\sum_{i=1}^n |x_i b_{\pi(i)}|^2\right)^{p/2}\right]^{1/p} \le$$

$$8^{1/p}\left(1+\frac{2}{n-1}\right)^{1/p}\|(|x_1|^p,\ldots,|x_n|^p)\|_{N_{b^p}}^{1/p},$$

where the Orlicz function N_{b^p} is defined by

$$\frac{1}{5}N_{b^p}^{*-1}\left(\frac{l}{n}\right)\le\frac{1}{n}\left\{\sum_{1\le i\le l}b_i^p+l^{\frac{2-p}{2}}\left(\sum_{l+1\le i\le n}b_i^2\right)^{\frac{p}{2}}\right\}\le 2N_{b^p}^{*-1}\left(\frac{l+1}{n}\right).$$

Thus we get

$$\frac{\pi_p(K_b)}{8^{1/p}\left(1+\frac{2}{n-1}\right)^{1/p}B_p}\le\sup_{x\neq 0}\frac{\|x\|_b}{\|(|x_1|^p,\ldots,|x_n|^p)\|_{N_{b^p}}^{1/p}}\le\frac{4^{1/p}A_p\pi_p(K_b)}{\left(\frac{1}{2}-\frac{1}{n-1}\right)^{1/p}}.$$

If $p=1$, this can be simplified as then by definition of the $\|\cdot\|_b$-norm

$$\sup_{x\neq 0}\frac{\|x\|_b}{\|(|x_1|^p,\ldots,|x_n|^p)\|_{N_{b^p}}^{1/p}}=\sup_{\|(|x_1|,\ldots,|x_n|)\|_{N_b}=1}\langle b,|x|\rangle=\|b\|_{N_b^*},$$

where $|x|$ denotes $\{|x_i|\}_i$. □

Using Lemma 5.11 we obtain the following result of Gluskin [G2] and independently Schütt [S].

Corollary 5.13 *There is an absolute constant $c>0$ such that*

(i) for every $\sqrt{n}\le k\le n$ one has

$$c\sqrt{n}\le\lambda(B_k)=n/\pi_1(B_k)<\sqrt{n};$$

(ii) for every $1\le k\le\sqrt{n}$ one has

$$ck\le\lambda(B_k)=n/\pi_1(B_k)\le k.$$

Proof: It is well known ([GG]) that for a symmetric space $(\mathbb{R}^n,K)$ one has $\lambda(K)\pi_1(K)=n$. Clearly, $(\mathbb{R}^n,B_k)$ is a symmetric space. This shows the equality. The upper estimates hold, since $\lambda(K)\le\sqrt{n}$ for every $K\subset\mathbb{R}^n$ and $\lambda(B_k)\le d(B_k,B_\infty^n)\le k$. It remains to prove the lower estimates.

Take $b\in\mathbb{R}^n$ such that $b_1=b_2=\ldots=b_k=1$, $b_{k+1}=\ldots=b_n=0$. Then $B_k=K_b$ and $\|\cdot\|_k=\|\cdot\|_b$. Clearly,

$$\frac{1}{n}\sum_{j=1}^n s(j)\ge\frac{k}{2n}\sum_{i\le n/k}|x_i|^*$$

and

$$\left(\frac{1}{n}\sum_{j=n+1}^{n^2} s(j)^2\right)^{1/2} \geq \left(\frac{k}{n}\sum_{i>n/k} (|x_i|^*)^2\right)^{1/2}.$$

Thus by Lemma 5.11 (applied for $p=1$) and Lemma 5.7 (applied for $q=2$) we have

$$\lambda(B_k) \geq \frac{n}{20}\cdot\min_{x\neq 0}\frac{(k/n)\sum_{i\leq n/k}|x_i|^* + \left((k/n)\sum_{i>n/k}(|x_i|^*)^2\right)^{1/2}}{\|x\|_k} =$$

$$\frac{k}{20}\cdot\min_{x\neq 0}\frac{\sum_{i\leq n/k}|x_i|^* + \left((n/k)\sum_{i>n/k}(|x_i|^*)^2\right)^{1/2}}{\sum_{i\leq k}|x_i|^*}.$$

Now, if $k \leq \sqrt{n}$ then

$$\lambda(B_k) \geq \frac{k}{20}\cdot\min_{x\neq 0}\frac{\sum_{i\leq n/k}|x_i|^*}{\sum_{i\leq k}|x_i|^*} \geq \frac{k}{20}.$$

If $k \geq \sqrt{n}$ then

$$\lambda(B_k) \geq \frac{k}{20}\cdot\min_{x\neq 0}\frac{\sum_{i\leq n/k}|x_i|^* + \left((n/k)\sum_{k\geq i>n/k}(|x_i|^*)^2\right)^{1/2}}{\sum_{i\leq k}|x_i|^*} \geq$$

$$\frac{k}{20}\cdot\min_{x\neq 0}\frac{\sum_{i\leq n/k}|x_i|^* + (\sqrt{n}/k)\sum_{k\geq i>n/k}|x_i|^*}{\sum_{i\leq k}|x_i|^*} \geq \frac{k}{20}\cdot\frac{\sqrt{n}}{k}.$$

That completes the proof. □

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Y. Gordon, Department of Mathematics, Technion, Haifa 32000, Israel.
e-mail: gordon@techunix.technion.ac.il

A.E. Litvak, Department of Mathematics, Technion, Haifa, Israel.
e-mail: alex@math.technion.ac.il
and
Department of Mathematical and Statistical Sciences, University of Alberta, AB, Canada, T6G 2G1.
e-mail: alexandr@math.ualberta.ca

C. Schütt, Christian Albrechts Universität, Mathematisches Seminar, 24098 Kiel, Germany.
e-mail: schuett@math.uni-kiel.de
and
Department of Mathematics, Case Western Reserve University, Cleveland, Ohio 44106, U. S. A.

E. Werner, Department of Mathematics, Case Western Reserve University, Cleveland, Ohio 44106, U. S. A.
e-mail: emw2@po.cwru.edu
and
Université de Lille 1, UFR de Mathématique, 59655 Villeneuve d'Ascq, France.