Diameters of Sections and Coverings of Convex Bodies *

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Abstract

We study the diameters of sections of convex bodies in $\mathbb{R}^N$ determined by a random $N \times n$ matrix Γ, either as kernels of Γ^* or as images of Γ. Entries of Γ are independent random variables satisfying some boundedness conditions, and typical examples are matrices with Gaussian or Bernoulli random variables. We show that if a symmetric convex body K in $\mathbb{R}^N$ has one well bounded k-codimensional section, then for any $m > ck$ random sections of K of codimension m are also well bounded, where $c \geq 1$ is an absolute constant. It is noteworthy that in the Gaussian case, when Γ determines randomness in sense of the Haar measure on the Grassmann manifold, we can take $c = 1$.

0 Introduction

Geometric Functional Analysis and the theory of finite dimensional normed spaces, traditionally study the structure of subspaces and quotient spaces of finite dimensional normed spaces, and operators acting on them. A parallel study in the general setting of convex bodies and in the language of Asymptotic Convex Geometry is concerned with the asymptotic properties of sections and projections of N-dimensional convex bodies, when N grows to infinity. Then n-dimensional subspaces (sections) or quotients (projections)

*Keywords: symmetric convex bodies, sections, random sections, diameters, covering numbers, Gelfand numbers.

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are often constructed using various random methods. Randomness may be understood in sense of the rotation invariant probability measure on the Grassmann manifold of n-dimensional subspaces of $\mathbb{R}^N$, or, for different purposes, it may carry some specific structure, such as in the case of subspaces generated by n vectors with random ± 1 coordinates in $\mathbb{R}^N$. These examples can be seen as particular cases of a general setting when randomness is determined by rectangular matrices with random variable entries.

Consider a "random" $N \times n$ matrix Γ acting as a mapping $\Gamma : \mathbb{R}^n \to \mathbb{R}^N$ (with $N \geq n$). We can adopt two points of view. Subspaces of $\mathbb{R}^N$ may be defined by n linear forms, with say, ± 1 coefficients, in which case we look for $\ker \Gamma^*$ of a random ± 1 matrix Γ. Alternatively, they may be generated by n vectors in $\mathbb{R}^N$, with say, ± 1 coordinates, and then we look for the image of $\mathbb{R}^n$ under a ± 1 matrix Γ. When the matrix Γ is Gaussian, it is rotation invariant from both sides, and then the induced measures on the linear subspaces, either $\ker \Gamma^*$ or $\Gamma(\mathbb{R}^n)$, are both the Haar measures on the Grassmann manifolds, but in general these measures may be different. Studies of random subspaces defined by linear forms have been quite extensive in the Gaussian case; and were developed in [MiP2] in the case of ± 1 coefficients and in a more general setting. The second approach, which is a dual point of view, is technically very different and appeared recently in [LPRT]. (One important difference between the present paper and the work in [LPRT], though, is that here we are more interested in properties of the subspaces $\Gamma(\mathbb{R}^n)$ than in properties of Γ itself.)

Let us recall a well known example. In the studies of Euclidean subspaces of ℓ_1^N, Kashin ([Ka]) proved that for every proportion $0 < \lambda < 1$, there exist subspaces of $\mathbb{R}^N$ of dimension $n = [\lambda N]$ on which the ℓ_1^N norm, defined for $x = (x_i) \in \mathbb{R}^N$ by $\|x\|_1 = \sum_1^N |x_i|$, and the Euclidean norm are equivalent (with constants independent on the dimension). In fact, "random" n-dimensional subspaces of $\mathbb{R}^N$ (in sense of the Haar measure on the Grassman manifold $G_{N,n}$) are "nearly" Euclidean in that sense. Kashin's theorem was reproved by Szarek [Sz] by a different argument, which also worked in a more general case of spaces with so-called bounded volume ratio [SzT] (cf. also [P]). If one asks for an additional structure on the subspace, for instance for subspaces defined by linear forms with random ± 1 coefficients, an analogous fact was proved in [MiP2]. For spaces generated by vectors with ± 1 coordinates, the problem was left open for some time. For ℓ_1^N, it was proved in [JS] (see [S]) that one such a subspace exists, and in [LPRTV1], [LPRTV2] that subspaces generated by random vectors with ± 1 coordinates satisfy the conclusion of

Kashin's result.

This example raises several questions. If a convex body K contains *one* k-codimensional section with a well bounded (Euclidean) diameter, does a *random* section of a slightly larger codimension μk also has a well bounded diameter? Randomness may be determined by the Haar measure on the Grassmann manifold, or by vectors with random ± 1 coordinates, or even by random matrices from a more general class. Can one take μ arbitrarily close to 1, in any of the above cases? Let us note that, since subspaces determined by random matrices from some class carry an additional structure associated to the class, therefore it is not even clear that the existence of one section of K with good control of the diameter yields the existence of even *just one* section of K determined by a matrix from the class and admitting a control of the diameter. In this paper we solve in positive the former series of questions, and we answer the latter question in the rotation invariant case.

This type of general phenomenon, of a *deterministic* information implying a *randomized* one, has been first observed in [MiS] in the context of what is called the "global" form of Dvoretzky's theorem. In a "local" context similar questions on the Grassmann manifold were considered in [LT], [MT1], [MT2]. In particular, the affirmative answer to the first question of the preceding paragraph immediately follows from one of the main results of [LT] (Theorem 3.2), and a version of this question for dimensions rather than codimensions was proved in [MT2] (Proposition 2.3). Independently, these questions, again for the Grassmann manifold, were introduced and answered in [GMT] and [V], however these proofs could not give μ close to 1. The approach in [GMT] and [V] is based on a recent Gromov's theorem on the sphere – the isoperimetric inequality for waists. In contrast, our approach is completely different and allows for a natural generalization to random structures determined by random matrices described above.

The paper is organized as follows. In Section 1, we give our main new technical tool on covering numbers, Theorem 1.3 and its Corollary 1.6, which will be important for the next sections. In Section 2, we study subspaces defined by linear forms, and we begin by illustrating this approach by the rotation invariant case. We give an estimate of the diameter of random sections, with respect to the Haar measure on the Grassmann manifold, assuming an information on the diameter of one section of higher dimension. Our main result, Theorem 2.4, states the following:
Let $1 \leq k < m < N$ be integers and $a > 0$, and set $\mu := m/k$. Let $K \subset \mathbb{R}^N$ be

an arbitrary symmetric convex body such that there exists a k-codimensional subspace E_0 with $\operatorname{diam}(E_0 \cap K) \le 2a$. *Then the subset of the Grassmann manifold of all m-codimensional subspaces $E \subset \mathbb{R}^N$ satisfying*

$$\operatorname{diam}(E \cap K) \le a \left(C \sqrt{N/m} \right)^{1+1/(\mu-1)},$$

has Haar measure larger than $1 - 2\,e^{-m/2}$ (here $C > 1$ is a universal constant).

The fact that there is almost no loss in the dimension, that is, $\mu = m/k$ can be chosen as close to 1 as we wish, resolves the question left open by all the previous proofs.

At the end of Section 2 we observe that our proof also works for subspaces given as the kernels of random $m \times N$ matrices from a class of matrices endowed with probability satisfying two abstract conditions. In particular, it gives a version of Theorem 2.4 for kernels of ± 1 random matrices. This general approach is further elaborated in the subsequent section.

In Section 3, we study the problem for sections defined by images by a random matrix Γ, when Γ carries some additional structure. We discuss general conditions on the set of random matrices that imply a similar statement as above; it turns out that, for example, the class of matrices with independent subgaussian entries satisfies these conditions. In Theorem 3.4 we prove estimates of the diameter of random sections – given as images under Γ and with respect to our new probability measure on the set of matrices – starting from an information on the diameter of one section of a higher dimension.

Let us finish by recalling basic notations used throughout the paper. If X is a normed space, we denote its unit ball by B_X. We denote the Euclidean norm on $\mathbb{R}^n$ by $|\cdot|$ and the Euclidean unit ball by B_2^n. By a convex body $K \subset \mathbb{R}^n$ we mean a convex compact set with the non-empty interior. We call K symmetric if it is centrally symmetric.

1 Results on covering numbers

In this section, we prove a result on covering numbers valid for operators between normed spaces. A setting from convex geometry is discussed at the end of the section. We start by recalling a few classical notations for operators.

If X and Y are normed spaces, an operator $u : X \to Y$ always means a bounded linear operator, with the operator norm denoted by $\|u\|$. For

$k = 1, 2, \ldots$, we denote by $a_k(u)$, $c_k(u)$ and $d_k(u)$ the approximation, Gelfand, and Kolmogorov numbers, respectively. Namely,

$$a_k(u) = \inf \|u - v\|,$$

where the infimum runs over all operators $v : X \to Y$ with rank $v < k$; then

$$c_k(u) = \inf \|u_{|E}\|,$$

where the infimum runs over all subspaces $E \subset X$ of codim $E < k$; and

$$d_k(u) = \inf \|Q_F u\|,$$

where the infimum runs over all subspaces $F \subset Y$ of $\dim F < k$ and $Q_F : Y \to Y/F$ denotes the quotient map.

Let X be a linear space and K, L be subsets of X. We recall that the covering number $N(K, L)$ is defined as the minimal number N such that there exist vectors x_1, ..., x_N in X satisfying

$$K \subset \bigcup_{i=1}^{N} (x_i + L). \tag{1.1}$$

Let $\varepsilon > 0$, a set of points x_1, ..., x_N in X satisfying $K \subset \cup_{i=1}^{N} (x_i + \varepsilon L)$ is called an ε-net of K with respect to L.

It is sometimes useful to specify a membership condition on the net (x_i): if $K \subset E \subset X$ and in the definition (1.1) we additionally require that $x_i \in E$, $1 \leq i \leq N$, then we shall use the notation $N_E(K, L)$ instead of $N(K, L)$. We also let $\bar{N}(K, L) = N_K(K, L)$.

For a convex body $L \subset \mathbb{R}^m$ and $\eta \in (0, 1)$, we shall often need an upper estimate for the covering number $N(L, \eta L)$. We could use a standard estimate by $(1 + 2/\eta)^m$, which follows by comparing volumes and which would be sufficient for the results in Section 2. However, we prefer to use here a more sophisticated estimate by Rogers-Zong ([RZ]), which leads to better results in this section.

Let $m \geq 1$, we set $\theta_m = \sup \theta(K)$, where the supremum is taken over all convex bodies $K \subset \mathbb{R}^m$ and $\theta(K)$ is the covering density of K (see [R2] for the definition and more details). It is known (see [R1], [R2]) that $\theta_1 = 1$, $\theta_2 \leq 1.5$, and, by a result of Rogers,

$$\theta_m \leq \inf_{0 < x < 1/m} (1 + x)^m (1 - m \ln x) < m(\ln m + \ln(\ln m) + 5)$$

for $m \geq 3$. The following lemma has been proved in [RZ].

Lemma 1.1 *Let K and L be two convex bodies in $\mathbb{R}^m$. Then*

$$N(K,L) \leq \theta_m \frac{|K-L|}{|L|}.$$

Our result on covering numbers is based on the following key proposition.

Proposition 1.2 *Let X, Y be normed spaces and let $u : X \to Y$ be an operator. Let $k \geq 1$ and $a > 0$, and let $w, v : X \to Y$ be such that $u = w + v$,* rank *$v \leq k$ and $\|w\| \leq a$. Let $E = vX$. Then for every $r > a$, we have*

$$N(uB_X, rB_Y) \leq N_E(uB_X, rB_Y) \leq \theta_k \left(\frac{\|u\| + r}{r - a} \right)^k .$$

Proof: Clearly, $\|v\| \leq \|u\| + a$. Therefore we have

$$u(B_X) \subset w(B_X) + v(B_X) \subset aB_Y + (\|u\| + a)B_Y \cap E. \tag{1.2}$$

Set $\varepsilon = r - a > 0$. Since $\dim E \leq k$, by Lemma 1.1 we can cover $(\|u\| + a)B_Y \cap E$ by $N \leq \theta_k \left(1 + (\|u\| + a)/\varepsilon\right)^k$ shifts (by vectors from E) of the balls $\varepsilon B_Y \cap E$, i.e.

$$(\|u\| + a)B_Y \cap E \subset \bigcup_{i=1}^{N} (x_i + \varepsilon B_Y \cap E),$$

where $x_i \in E$, $1 \leq i \leq N$. Then the latter set in (1.2) is contained in

$$aB_Y + \bigcup_{i=1}^{N} (x_i + \varepsilon B_Y) \subset \bigcup_{i=1}^{N} (x_i + (a + \varepsilon)B_Y).$$

Since $r = a + \varepsilon$, we get $N_E(u(B_X), rB_Y) \leq N \leq \theta_k \left(1 + (\|u\| + a)/(r - a)\right)^k$, which implies the desired result. □

As a consequence we obtain the following theorem.

Theorem 1.3 *Let X, Y be normed spaces and let $u : X \to Y$. Let $k \geq 1$ and $a > 0$ satisfy $a_{k+1}(u) \leq a$. Then for every $r > a$, one has*

$$N(uB_X, rB_Y) \leq \theta_k \left(\frac{\|u\| + r}{r - a} \right)^k .$$

Proof: This theorem immediately follows from Proposition 1.2 and the definition of $a_k(u)$. $\square$

The next proposition is standard (see e.g. [Pi], also [P], Proposition 5.1, for definitions and details).

Proposition 1.4 *Let X, Y be normed spaces and let $u : X \to Y$. Assume that $j : Y \to \ell_\infty(I)$ is an isometric embedding and that $Q : \ell_1(J) \to X$ is a quotient map, where I and J are some sets of indexes. Then for every $\varepsilon > 0$ and every $k \geq 1$ we have*

(i) $\bar{N}(uB_X, 2\varepsilon B_Y) \leq N(juB_X, \varepsilon B_{\ell_\infty}) \leq N(uB_X, \varepsilon B_Y)$ *and* $c_k(u) = a_k(ju)$*;*

(ii) $N(uB_X, \varepsilon B_Y) = N(uQB_{\ell_1}, \varepsilon B_Y)$ *and* $d_k(u) = a_k(uQ)$*.*

Theorem 1.3 together with Proposition 1.4 immediately imply the following corollary.

Corollary 1.5 *Let X, Y be normed spaces and let $u : X \to Y$. Let $k \geq 1$ and $a > 0$, and assume that either $c_{k+1}(u) \leq a$ or $d_{k+1}(u) \leq a$. Then for every $r > a$ one has*

$$\bar{N}(uB_X, 2rB_Y) \leq \theta_k \left(\frac{\|u\| + r}{r - a} \right)^k .$$

In geometric setting we typically consider convex bodies in $\mathbb{R}^N$ and the identity operators. Let us state a particular case of Corollary 1.5 that we will use later. Let $K, L \subset \mathbb{R}^N$ be symmetric convex bodies. Let X be $\mathbb{R}^N$ equipped with the norm for which the unit ball is $B_X = K$ and let Y be $\mathbb{R}^N$ equipped with the norm for which $B_Y = L$. Applying Corollary 1.5 to the identity operator we get the following:

Corollary 1.6 *Let $K, L \subset \mathbb{R}^N$ be symmetric convex bodies. Let $k \geq 1$ and $A > a > 0$ such that $K \subset AL$ and $K \cap E \subset aL$ for some k-codimensional subspace E of $\mathbb{R}^N$. Then for every $r > a$ one has*

$$N(K, 2rL) \leq \theta_k \left(\frac{A + r}{r - a} \right)^k .$$

An analogous statement using now Kolmogorov information is also valid:

Corollary 1.7 *Let $K, L \subset \mathbb{R}^N$ be symmetric convex bodies. Let $k \geq 1$ and let $A > a > 0$ such that $K \subset AL$ and $PK \subset aPL$ for some projection P of corank k. Then for every $r > a$ one has*

$$\bar{N}(K, rL) \leq \theta_k \left(\frac{A + r}{r - a} \right)^k .$$

A slightly weaker form of Corollary 1.7 was proved by Rudelson and Vershynin ([V]) in the case when K is the Euclidean ball.

2 Diameters of random sections

We study the diameters of random sections of a convex body, where the sections are given by kernels of a random Gaussian matrix, with the induced natural measure. On the one hand, the results can be reformulated for sections viewed as elements of the Grassman manifold. On the other hand, this approach may be developed for a larger class of random matrices, as will be observed at the end of this section.

Let $1 \leq m \leq N$, and let

$$G : \mathbb{R}^N \to \mathbb{R}^m$$

be a random $m \times N$ matrix with independent $N(0, 1/N)$ distributed Gaussian entries.

For future reference, we recall a well-known estimate ([DS], Theorem 2.13). Let $\beta = m/N$, then for every $t > 0$, we have

$$\mathbb{P} \left(\|G : \ell_2^N \to \ell_2^m\| > 1 + \sqrt{\beta} + t \right) \leq e^{-Nt^2/2}.$$

In particular,

$$\mathbb{P} \left(\|G : \ell_2^N \to \ell_2^m\| > 1 + 2\sqrt{m/N} \right) \leq e^{-m/2}. \tag{2.1}$$

For $\xi \in [0, 1]$, let

$$p(\xi) := \mathbb{P}\big\{ |Gx_0| \leq \xi |x_0| \big\},$$

where $x_0 \in \mathbb{R}^N$ is an arbitrary vector (note that this probability does not depend on x_0).

We estimate the probability $p(\xi)$ by a direct calculation, which may be of independent interest. (In fact, a weaker estimate $p(\xi) \le |\xi\sqrt{N} B_2^m| = (\xi\sqrt{N})^m v_m$, where v_m is defined and estimated in (2.2) below, would be sufficient for Theorem 2.4.)

Lemma 2.1 *Let $A > 1$ and $1 \le m \le N$. Let $G : \mathbb{R}^N \to \mathbb{R}^m$ be a Gaussian matrix normalized as above. Then for every $0 < \xi < \sqrt{(A-1)m/(AN)}$ we have*

$$p(\xi) \le \frac{A}{\sqrt{\pi m}} \left(e\, \xi^2 \, \frac{N}{m} \right)^{m/2} \exp\left(-\xi^2 N/2\right).$$

Proof: Let (g_i) be independent $N(0,1)$-distributed Gaussian random variables. We have

$$\begin{aligned} p(\xi) = \mathbb{P}\big\{ |Gx_0| \le \xi |x_0| \big\} &= \mathbb{P}\left\{ \sum_{i=1}^m g_i^2 \le \xi^2 N \right\} \\ &= (2\pi)^{-m/2} \int_{\xi\sqrt{N} B_2^m} \exp(-|x|^2/2) dx. \end{aligned}$$

Thus to get the desired estimate it is sufficient to estimate the latter integral.

Let

$$v_m := |B_2^m| = \frac{\pi^{m/2}}{\Gamma(1+m/2)} \le \frac{1}{\sqrt{\pi m}} \left(\frac{2e\pi}{m} \right)^{m/2}. \tag{2.2}$$

Then for every $a \le \sqrt{(A-1)m/A}$, we have

$$\begin{aligned} \int_{aB_2^m} \exp(-|x|^2/2) dx &= v_m \int_0^a m t^{m-1} \exp(-t^2/2) dt \\ &\le A v_m \int_0^a \left(m t^{m-1} \exp(-t^2/2) - t^{m+1} \exp(-t^2/2) \right) dt \\ &= A v_m t^m \exp(-t^2/2) \,\Big|_0^a \le A \exp(-a^2/2) \frac{1}{\sqrt{\pi m}} \left(\frac{2e\pi a^2}{m} \right)^{m/2}. \end{aligned}$$

This concludes the proof. □

Now we pass to the main subject of this section and we start by introducing some convenient additional notation. A set K is called star-shaped if $tK \subset K$ for all $0 \le t \le 1$. For a set $K \subset \mathbb{R}^N$ we denote by diam K the diameter of the Euclidean ball centered at 0 and circumscribed on K.

For any $\rho > 0$ we set

$$K_\rho = K \cap \rho B_2^N. \tag{2.3}$$

Recall a well-known and elementary fact:

Fact 2.2 *Let $K \subset \mathbb{R}^N$ be star-shaped. If $E \subset \mathbb{R}^N$ is a subspace such that* $\operatorname{diam}(E \cap K_\rho) < 2\rho$, *then* $\operatorname{diam}(E \cap K) = \operatorname{diam}(E \cap K_\rho) < 2\rho$.

We also set, for $\rho > 0$ and any $\varepsilon > 0$,

$$N_\rho(\varepsilon) := N(K_\rho, \varepsilon B_2^N). \tag{2.4}$$

The following probabilistic estimate is a starting point for our result.

Proposition 2.3 *Let $1 \le m < N$ be positive integers, $\varepsilon > 0$ and $\xi \in [0,1]$. Let $K \subset \mathbb{R}^N$ be star-shaped. Then*

$$\mathbb{P}\{\operatorname{diam}\big(\ker G \cap K\big) < 2\rho\} \ge 1 - N_\rho(\varepsilon) p(\xi) - e^{-m/2}, \tag{2.5}$$

where $\rho = 4\,\varepsilon/\xi$.

Proof: Let Λ be an ε-net of K_ρ with respect to the Euclidean norm, satisfying $|\Lambda| \le N_\rho(\varepsilon)$. Let $x \in K_\rho$ and let $x_0 \in \Lambda$ such that $|x - x_0| \le \varepsilon$. Suppose that $x \in \ker G$ and $|Gx_0| > \xi |x_0|$.

Using (2.1) we get that, with probability $\ge 1 - e^{-m/2}$, we have

$$\begin{aligned} |x| &\le |x - x_0| + |x_0| < \varepsilon + \frac{|Gx_0|}{\xi} = \varepsilon + \frac{|G(x_0 - x)|}{\xi} \\ &\le \varepsilon\big(1 + \|G\|/\xi\big) \le \varepsilon\left(1 + (1 + 2\sqrt{m/N})/\xi\right) \\ &\le (4\,\varepsilon/\xi) = \rho, \end{aligned}$$

for any $0 < \xi \le 1$. Since the probability that $|Gx_0| > \xi|x_0|$ for all $x_0 \in \Lambda$ is larger than or equal to $1 - N_\rho(\varepsilon)p(\xi)$, then

$$\mathbb{P}\left\{\operatorname{diam}\big(\ker G \cap K_\rho\big) < 2\rho\right\} \ge 1 - N_\rho(\varepsilon)p(\xi) - e^{-m/2}.$$

Now, by Fact 2.2, the latter estimate immediately implies (2.5). $\square$

The main result of this section gives an estimate of the diameter of random sections of symmetric convex bodies with respect to the rotation invariant probability measure on the Grassmann manifold, assuming an information on the minimal diameter of sections of a slightly smaller codimension. By allowing the ratio of the codimensions of minimal sections and of random sections to be arbitrarily close to 1, it resolves the case left open by all the previous proofs; in particular the result in [GMT] is valid for $\mu > 2$ and in [V] for $\mu > 32$. It is worthwhile to note that in the work [GMT] the only obstruction for letting μ close to 1, is due to the use of an isoperimetric inequality of Gromov, which is still open in an arbitrary dimension. Our work also provides a more elementary proof of results from these papers in the full range of dimensions (hence also codimensions).

Theorem 2.4 *Let $1 \leq k < m < N$ be integers and $a > 0$, and set $\mu := m/k$. Let $K \subset \mathbb{R}^N$ be a symmetric convex body such that there exists a k-codimensional subspace E_0 with $\operatorname{diam}(E_0 \cap K) \leq 2a$. Then the subset of the Grassmann manifold $G_{N,N-m}$ of all m-codimensional subspaces $E \subset \mathbb{R}^N$ satisfying*

$$\operatorname{diam}(E \cap K) \leq a\,\left(C\,\sqrt{N/m}\right)^{1+1/(\mu-1)},$$

has Haar measure larger than $1 - 2\,e^{-m/2}$ (here $1 < C < 100$ is a universal constant).

Proof: Instead of estimating the measure of the subset of $G_{N,N-m}$ considered in the theorem we shall prove an analogous estimate for the probability

$$\mathbb{P}\left\{\operatorname{diam}(\ker G \cap K) \leq a\,\left(C\,\sqrt{N/m}\right)^{1+1/(\mu-1)}\right\}, \tag{2.6}$$

where $G : \mathbb{R}^N \to \mathbb{R}^m$ is a Gaussian random matrix.

Let $\varepsilon := 4a$. Fix $\xi < \sqrt{m/N}/e$ to be determined later, and set $\rho := 4\varepsilon/\xi = 16a/\xi$. By Corollary 1.6, we have

$$N_\rho(\varepsilon) \leq \theta_k \left(\frac{2\rho + \varepsilon}{\varepsilon - 2a}\right)^k = \theta_k \big(2(1 + 8/\xi)\big)^k \leq \theta_k\,(18/\xi)^k.$$

Therefore applying Lemma 2.1 with $A = 5/4$ we obtain

$$N_\rho(\varepsilon)\,p(\xi) \leq \theta_k\,(18/\xi)^k\,(5/4)\,\left(\xi\sqrt{e\,N/m}\right)^m.$$

An elementary calculation shows that the latter expression is less than or equal to $e^{-m/2}$ whenever

$$\xi \le \xi_0 := \left(e\sqrt{N/m}\right)^{-m/(m-k)} 18^{-k/(m-k)} \big(4/(5\theta_k)\big)^{1/(m-k)}.$$

Letting $\xi = \xi_0$ we get

$$\begin{aligned} \rho = 16a/\xi &\le 16a \left(e\sqrt{N/m}\right)^{\mu/(\mu-1)} \left(18(5\theta_k/4)^{1/k}\right)^{1/(\mu-1)} \\ &\le a\left(C\sqrt{N/m}\right)^{1+1/(\mu-1)}, \end{aligned}$$

for a certain absolute constant C. Combining this estimate with Proposition 2.3 and the probability estimate above, we conclude the proof. $\square$

The previous result gives an estimate of the diameter of random sections of a symmetric convex body K, when randomness is on the Grassmann manifold, or equivalently, when random sections are generated by kernels of a Gaussian matrix. The method of proof extends to a large class of matrices. More precisely, we can consider random sections generated by kernels of a random matrix, satisfying certain natural conditions required for the proof to work. We shall outline the new framework but we omit the details of the proofs which are very similar to those discussed above.

Let $1 \le m < N$, we consider a set $\bar{\mathrm{M}}_{m,N}$ of $m \times N$ matrices T (treated as operators $T : \mathbb{R}^N \to \mathbb{R}^m$), endowed with a probability measure $\mathbb{P}$. We say that $\bar{\mathrm{M}}_{m,N}$ satisfies conditions *($\bar{M}$1)* and *($\bar{M}$2)* whenever

($\bar{M}$1) there exist $0 < \bar{t}_0 < 1$ and $0 < \bar{\nu}_0 < 1$ such that for every $x_0 \in \mathbb{R}^N$ we have

$$\mathbb{P}\left\{|Tx_0| \le \bar{t}_0 |x_0|\right\} \le \bar{\nu}_0;$$

($\bar{M}$2) there exist $\bar{a}_1 > 0$ and $0 < \bar{\nu}_1 < 1$ such that

$$\mathbb{P}\left\{\|T : \ell_2^N \to \ell_2^m\| > \bar{a}_1\right\} \le \bar{\nu}_1.$$

Of course the set of Gaussian random $m \times N$ matrices with independent $N(0, 1/N)$ distributed entries satisfies conditions *($\bar{M}$1)* (for an arbitrary $0 < \bar{t}_0 \le 1$, and a suitable $\bar{\nu}_0$) and *($\bar{M}$2)*; this follows from (2.1) and Lemma 2.1.

In fact, these conditions are satisfied for a large class of matrices T, whose transposed, $\Gamma = T^*$, belong to the class $\mathrm{M}'(N, m, b, a_1, a_2)$ (for some $b \geq 1$ and $a_1, a_2 > 0$), defined by (3.10) and (3.11) below. Condition *($\bar{M}$1)* which is the one non-trivial to check, follows from Proposition 3.4 in [LPRT]. Namely, the inspection of the proof in [LPRT] shows that the argument works for any rectangular random matrix satisfying the moment conditions, but without any relation between the number of rows and the number of columns. More precisely, taking into account our normalization, a matrix $T = (\xi_{ji})_{1 \leq j \leq m, 1 \leq i \leq N}$ satisfying (3.10) admits an estimate, for every $x_0 \in \mathbb{R}^N$,

$$\mathbb{P}\left\{|Tx_0| \leq c' b^{-3}\sqrt{m/N}\,|x_0|\right\} \leq e^{-c'' m/b^6}, \tag{2.7}$$

where $0 < c', c'' < 1$ are universal constants. In particular, all examples of matrices which work in Section 3, such as matrices with independent $\pm 1/\sqrt{N}$ entries and, more generally, subgaussian matrices, work as well here (after transposition).

Just repeating the proof of Proposition 2.3 we get the following:

Proposition 2.3' *Let $1 \leq m < N$ and $\varepsilon > 0$. Let $\bar{\mathrm{M}}_{m,N}$ with a probability measure $\mathbb{P}$ satisfy conditions ($\bar{M}$1) and ($\bar{M}$2). Let $K \subset \mathbb{R}^N$. Then*

$$\mathbb{P}\left\{\mathrm{diam}\left(\ker T \cap K\right) < 2\rho\right\} \geq 1 - N_\rho(\varepsilon)\bar{\nu}_0 - \bar{\nu}_1,$$

where $\rho = \varepsilon(1 + \bar{a}_1/\bar{t}_0)$.

Using this and the proof of Theorem 2.4, we get the general statement:

Theorem 2.4' *Let $1 \leq k < m < N$ be integers and $a > 0$. Let $\bar{\mathrm{M}}_{m,N}$ with a probability measure $\mathbb{P}$ satisfy conditions ($\bar{M}$1) and ($\bar{M}$2). Let*

$$\bar{M} := \theta_k\left(6 + 4\bar{a}_1/\bar{t}_0\right)^k \leq \theta_k\left(10\max(\bar{a}_1/\bar{t}_0, 1)\right)^k.$$

Let $K \subset \mathbb{R}^N$ be a symmetric convex body such that there exists a k-codimensional subspace E_0 with $\mathrm{diam}\,(E_0 \cap K) \leq 2a$. *Then*

$$\mathbb{P}\left\{\mathrm{diam}\left(\ker T \cap K\right) < 8\,a\,(1 + \bar{a}_1/\bar{t}_0)\right\} \geq 1 - \bar{M}\,\bar{\nu}_0 - \bar{\nu}_1. \tag{2.8}$$

As an example of an application we have the following result for sections determined by kernels of matrices with independent ± 1 entries.

Corollary 2.5 *Let $m < N$ and $a > 0$. Let T be an $m \times N$ matrix with independent ± 1 entries. Let $1 \le k < c\,m/\log(N/m)$. Let $K \subset \mathbb{R}^N$ be a symmetric convex body such that there exists a k-codimensional subspace E_0 with* $\operatorname{diam}(E_0 \cap K) \le 2a$. *Then*

$$\mathbb{P}\left\{\operatorname{diam}\left(\ker T \cap K\right) < C\,a\,\sqrt{N/m}\right\} \ge 1 - e^{-c'''m}. \tag{2.9}$$

Here $0 < c < 1$ and $C, c''' > 0$ are universal constants.

Sketch of the proof: Note that the matrix $T/\sqrt{N}$ satisfies conditions *($\bar{M}1$)* and *($\bar{M}2$)* with $\bar{t}_0 = c'\sqrt{m/N}$, $\bar{a}_1$ an absolute constant, and $\bar{\nu}_0 = \bar{\nu}_1 = e^{-c''m}$, where $c', c'' > 0$ are absolute constants. Therefore (2.9) follows from (2.8) provided that $\bar{M}\,\bar{\nu}_0 \le e^{-c''m/2}$. Similarly as in the proof of Theorem 3.4 below, this inequality is satisfied once we have

$$\left(C'\sqrt{N/m}\right)^{k/m} e^{-c''} \le e^{-c''/2}$$

which can be ensured by the assumption $k < c\,m/\log(N/m)$, for an appropriately chosen constant $c > 0$. □

3 Diameters via random embeddings

In this section, we shall consider rectangular $N \times n$ matrices (with $1 \le n < N$) acting as embeddings from $\mathbb{R}^n$ into $\mathbb{R}^N$. Accordingly, for $1 \le n < N$, we shall consider a set $\mathrm{M}_{N,n}$ of $N \times n$ matrices Γ, endowed with a probability measure $\mathbb{P}$. We say that $\mathrm{M}_{N,n}$ satisfies conditions *(M1)* and *(M2)* whenever

(M1) for some $0 < t_0 < 1$ and $0 < \nu_0 < 1$ we have

$$\mathbb{P}\left\{\exists\, x \in S_{n-1} \ \text{ s.t. } \ \Gamma x \in z + t_0 B_2^N\right\} \le \nu_0,$$

for every $z \in \mathbb{R}^N$;

(M2) for some $a_1 > 0$ and $0 < \nu_1 < 1$ we have

$$\mathbb{P}\left\{\|\Gamma : \ell_2^n \to \ell_2^N\| > a_1\right\} \le \nu_1.$$

It turns out that the above conditions are already sufficient to estimate the diameters of sections determined by $\Gamma(\mathbb{R}^n) \subset \mathbb{R}^N$, given by a "random" embedding Γ. This is shown in the following theorem, similar in character to Proposition 2.3 and Theorem 2.4.

Theorem 3.1 *Let $1 \le n < N$. Let $\mathrm{M}_{N,n}$ with a probability measure $\mathbb{P}$ satisfy conditions (M1) and (M2) (with $t_0 < 6a_1$). Let $k \ge 1$ satisfy*

$$M := \theta_k (6a_1/t_0)^k < (1-\nu_1)/\nu_0. \tag{3.1}$$

If $K \subset \mathbb{R}^N$ is a symmetric convex body, and for some $a > 0$ there exists a k-codimensional subspace $F \subset \mathbb{R}^N$ such that

$$\mathrm{diam}\,(K \cap F) \le 2a, \tag{3.2}$$

then

$$\mathbb{P}\left\{\mathrm{diam}\left(K \cap \Gamma(\mathbb{R}^n)\right) < 8\,a\,a_1/t_0\right\} \ge 1 - M\,\nu_0 - \nu_1. \tag{3.3}$$

Recall that for a set $K \subset \mathbb{R}^N$ and $\rho > 0$, the set K_ρ was defined in (2.3).

Proof: We first show that for an arbitrary $\rho > 0$, letting $\varepsilon := (t_0/a_1)\rho$, we have

$$\mathbb{P}\left\{\mathrm{diam}\left(K \cap \Gamma(\mathbb{R}^n)\right) < 2\rho\right\} \ge 1 - N_\rho(\varepsilon)\,\nu_0 - \nu_1, \tag{3.4}$$

where the notation of $N_\rho(\varepsilon) = N(K_\rho, \varepsilon B_2^N)$ was introduced in (2.4).

We start by considering the probability

$$\mathbb{P}\left\{\|\Gamma x\|_{K_\rho} > (a_1/\rho)|x| \text{ for all } x \in \mathbb{R}^n\right\}. \tag{3.5}$$

Observe that

$$N((a_1/\rho)K_\rho, t_0 B_2^N) = N_\rho(\varepsilon).$$

Let Λ be an t_0-net in $(a_1/\rho)K_\rho$ with respect to the Euclidean norm and such that $|\Lambda| \le N_\rho(\varepsilon)$. Then the complement of the set considered in (3.5) has probability

$$\begin{aligned}
&\mathbb{P}\{\ \exists\, x \in S_{n-1} \ \text{ s.t. } \ \Gamma x \in (a_1/\rho)K_\rho\ \}\\
&\le\ \mathbb{P}\left(\bigcup_{z\in\Lambda}\{\exists\, x \in S_{n-1} \ \text{ s.t. } \ \Gamma x \in z + t_0 B_2^N\}\right)\\
&\le\ N_\rho(\varepsilon)\,\nu_0.
\end{aligned}$$

Thus the probability in (3.5) is $\geq 1 - N_\rho(\varepsilon)\,\nu_0$.

To conclude the proof of (3.4) we note that the set considered in (3.5) is contained in the union

$$\Big\{\|\Gamma\| \leq a_1 \text{ and } \|\Gamma x\|_{K_\rho} > (a_1/\rho)|x| \text{ for all } x \in \mathbb{R}^n\Big\} \;\cup\; \big\{\|\Gamma\| > a_1\big\}$$
$$\subset\; \Big\{\|\Gamma x\|_{K_\rho} > (1/\rho)|\Gamma x| \text{ for all } x \in \mathbb{R}^n\Big\} \;\cup\; \big\{\|\Gamma\| > a_1\big\}.$$

By property *(M2)* this implies that, with probability $\geq 1 - N_\rho(\varepsilon)\,\nu_0 - \nu_1$, we have $\operatorname{diam}(K_\rho \cap \Gamma(\mathbb{R}^n)) < 2\rho$. Thus (3.4) follows by Fact 2.2.

Now returning to the proof of (3.3), set $\varepsilon = 4a$ and $\rho = \varepsilon a_1/t_0 = 4a\,a_1/t_0$. By Corollary 1.6, we immediately get

$$N_\rho(\varepsilon) \leq \theta_k \left(\frac{2\rho+\varepsilon}{\varepsilon - 2a}\right)^k = \theta_k\big(2(2a_1/t_0+1)\big)^k \leq \theta_k(6a_1/t_0)^k = M.$$

Therefore the proof of the theorem is concluded by applying (3.4). $\square$

Theorem 3.1 can be used to obtain a still different proof of estimates on the Grassmann manifold of the same type (and the same asymptotic order with respect to all parameters) as Theorem 2.4. In this case the proof is based on an estimate on the Grassmann manifold (or, equivalently, on the orthogonal group) which is considerably deeper than its counterpart in Section 2. The following proposition is of independent interest (and plays a similar role in the further argument as the estimate from Lemma 2.1).

Proposition 3.2 *Let $\delta > 0$, let $n \geq 1$ and $N = (1+\delta)n$. Let $\mu_{N,n}$ be the Haar measure on the Grassmann manifold $G_{N,n}$. For every $0 < t \leq 1/2$ and every $z \in \mathbb{R}^N$ we have the estimate*

$$\mu_{N,n}\big\{E \;:\; \exists\, x \in S_{N-1} \cap E \text{ s.t. } \; x \in z + tB_2^N\big\} \leq \left(4\,e\,t^2\,\frac{1+\delta}{\delta}\right)^{\delta n/2}. \quad (3.6)$$

Proof: We may assume that $z = re_1$ for some $r > 0$. Denote the subset of $G_{N,n}$ discussed in the statement by $\mathcal{F}_t$. If $\mathcal{F}_t$ is non-empty then we must have $|1-r| \leq t$, equivalently,

$$1 - t \leq r \leq 1 + t. \quad (3.7)$$

A direct two-dimensional calculation shows that for $x = (x_1, \ldots, x_N) \in S_{N-1}$ the condition $|x - re_1| \le t$ is equivalent to

$$x_1 \ge \frac{1 + r^2 - t^2}{2r} =: s.$$

Using this it is easy to see that $\mathcal{F}_t$ consists of all subspaces E for which there exists $x \in S_{N-1} \cap E$ such that $x_1 \ge s$; in other words, such that the (Euclidean) distance from x to the subspace spanned by e_1 satisfies $d\,(x, [e_1]) \le \sqrt{1 - s^2}$. Since for such E we have $d\,(e_1, E) \le d\,(e_1, [x]) = d\,(x, [e_1]) \le \sqrt{1 - s^2}$ (where $[x]$ is the subspace spanned by x), then

$$\mathcal{F}_t \subset \left\{ E \ : \ E \cap \left(e_1 + \sqrt{1 - s^2}\, B_2^N \right) \ne \emptyset \right\}.$$

The measure of this latter set can be estimated for example by the inequality from [MiP1], Lemma 6, which can be reformulated that for every $w \in \mathbb{R}^N$ and every $0 < \alpha < 1$, we have

$$\mu_{N,n} \left\{ E \ : \ E \cap (w + \alpha\, |w| B_2^N) \ne \emptyset \right\} \le \left(\alpha \sqrt{\frac{N\, e}{N - n}} \right)^{N-n}. \tag{3.8}$$

Therefore,

$$\mu_{N,n}(\mathcal{F}_t) \le \left((1 + \delta)\, e\, (1 - s^2)/\delta \right)^{\delta n/2}.$$

It is easy to check that $1 - s^2 \le 2(1 - s) \le 4t^2/(2r) \le 4t^2$ (the latter inequality follows from the fact that $t \le 1/2$ implies $r \ge 1/2$). Putting this into the last estimate we get $\mu_{N,n}(\mathcal{F}_t) \le (4\, e\, t^2\, (1 + \delta)/\delta\,)^{\delta n/2}$, as required. $\square$

Remark. Condition (3.6) can be equivalently expressed in terms of the orthogonal group $\mathcal{O}_N$. Namely, if μ_N denotes the Haar measure on $\mathcal{O}_N$ then, for every $0 < t \le 1/2$ and $z \in \mathbb{R}^N$,

$$\mu_N \left\{ U : \ \exists\, x \in S_{N-1} \cap \mathbb{R}^n \text{ s.t. } \ Ux \in z + tB_2^N \right\} \le \left(4\, e\, t^2\, \frac{1 + \delta}{\delta} \right)^{\delta n/2}. \tag{3.9}$$

Now, let again $\delta > 0$, $n \ge 1$ and $N = (1 + \delta)n$. Consider the set $M_{N,n}$ of $N \times n$ matrices whose n columns are orthonormal in $\mathbb{R}^N$, with the natural probability measure induced from the orthogonal group $\mathcal{O}_N$. Then (3.9) implies that $M_{N,n}$ satisfies condition *(M1)* for *every* $0 < t_0 \le 1/2$, with the corresponding $\nu_0 = \left(4\,e\,t_0^2\,(1+1/\delta)\right)^{\delta n/2}$. Since *(M2)* is obviously satisfied with $a_1 = 1$, for all $U \in M_{N,n}$, we are in position to apply Theorem 3.1 in a strong way.

Let $1 \le k < \delta n$ and set $\mu = \delta n/k$. Then it is easy to check that letting

$$\xi_0 := \left(\frac{1}{2\,e}\sqrt{\frac{\delta}{1+\delta}}\right)^{1+1/(\mu-1)} 18^{-1/(\mu-1)},$$

we get that for $t_0 = \xi_0$,

$$\theta_k \left(\frac{6}{t_0}\right)^k \left(4e\;t_0^2\;\frac{1+\delta}{\delta}\right)^{\delta n/2} \le e^{-\delta n/2}.$$

Thus, whenever for some $F \subset \mathbb{R}^N$ of codimension k, (3.2) is satisfied, then (3.3) holds, which translates into the same estimate on the Grassmann manifold as in Theorem 2.4.

As another application of Theorem 3.1 let us note that conditions *(M1)* and *(M2)* are in fact satisfied by a wide class of matrices, namely, the class $\mathrm{M}(N, n, b, a_1, a_2)$ considered in [LPRT], [LPRTV1] and [LPRTV2] (for some parameters b, a_1, a_2). In our setting it is more convenient to consider this class with a different normalization. For $1 \le n < N$, $b \ge 1$ and $a_1, a_2 > 0$, we define the set of $N \times n$ matrices $\mathrm{M}'(N, n, b, a_1, a_2)$ to consist of matrices Γ with real-valued independent symmetric random variable entries $(\xi_{ij})_{1\le i\le N, 1\le j\le n}$ satisfying:

$$1/\sqrt{N} \le \|\xi_{ij}\|_{L^2} \le \|\xi_{ij}\|_{L^3} \le b/\sqrt{N} \text{ for } 1 \le i \le N,\ 1 \le j \le n \quad (3.10)$$
$$\text{and} \quad \mathbb{P}\left(\|\Gamma : \ell_2^n \to \ell_2^N\| \ge a_1\right) \le e^{-a_2 N}. \quad (3.11)$$

Basic examples of matrices from $\mathrm{M}'(N, n, \mu, a_1, a_2)$ are random matrices whose entries are centered Gaussian of variance $1/N$ or symmetric $(\pm 1/\sqrt{N})$ random variables, and we will be mostly interested in the latter one. We refer the reader to [LPRT] for more information.

The fact that the set $\mathrm{M}'(N, n, b, a_1, a_2)$ satisfies *(M1)* is one of the main technical results in [LPRTV1], [LPRTV2]. Due to a technical form of dependencies of constants on the parameters, we state it as a separate lemma.

Lemma 3.3 *Let $\delta > 0$, $n > 1$ and $N = (1+\delta)n$, and let $b \ge 1$ and $a_1, a_2 > 0$. There exist $c_1 > 0$ of the form $c_1 = c_3^{1+1/\delta}$, and $\tilde{c}_1, c_2 > 0$ such that whenever*

$n \geq \tilde{c}_1^{1+1/\delta}$ then the set $\mathrm{M}'(N, n, b, a_1, a_2)$ satisfies (M1) with $t_0 = c_1$ and $\nu_0 = \exp(-c_2 N)$. Here $0 < c_3 < 1$ and $\tilde{c}_1$ depend on b and a_1, while $c_2 > 0$ depends on b and a_2 only.

Since $\mathrm{M}'(N, n, b, a_1, a_2)$ satisfies *(M2)* with $\nu_1 = \exp(-a_2 N)$, we get a result for random sections of bodies generated by columns of matrices from $\mathrm{M}'(N, n, b, a_1, a_2)$.

For the convenience of the proof we shall assume, as we clearly may without loss of generality, that $a_1 \geq 1$.

Theorem 3.4 *Let $\delta > 0$, $n > 1$ and $N = (1+\delta)n$, and let $a_1, b \geq 1$ and $a_2 > 0$. There exist $\mu_0 > 1$ such that for $1 \leq k \leq \delta n/\mu_0$ the following holds. Let Γ be a random matrix from $\mathrm{M}'(N, n, b, a_1, a_2)$. Let $K \subset \mathbb{R}^N$ be a symmetric convex body and assume that for some $a > 0$ there exists a k-codimensional subspace $F \subset \mathbb{R}^N$ such that*

$$\operatorname{diam}(K \cap F) \leq 2a.$$

There exist $\tilde{c}_1, c_2' > 0$ and $0 < c_3 < 1$ such that whenever $n \geq \tilde{c}_1^{1+1/\delta}$ then, with probability $\geq 1 - \exp(-c_2' N)$, we have

$$\operatorname{diam}(K \cap \Gamma(\mathbb{R}^n)) \leq 8\, a\, a_1 / c_3^{1+1/\delta}.$$

Here $0 < c_3 < 1$ and $\tilde{c}_1$ depend on b and a_1; $c_2' > 0$ depends on b and a_2; and μ_0 depends on b and a_1, a_2.

Proof: Let $\tilde{c}_1, c_2 > 0$ and $0 < c_3 < 1$ be from Lemma 3.3. We are looking for a bound for k which ensures that

$$\theta_k \left(6a_1/c_3^{1+1/\delta}\right)^k e^{-c_2 N} < e^{-c_2 N/2}. \tag{3.12}$$

Combining this condition with Lemma 3.3 and Theorem 3.1, we would get the required estimate on the diameter, with probability $\geq 1 - e^{-a_2 N} - e^{-c_2 N/2} \geq 1 - e^{-c_2' N}$, for an appropriate c_2'.

Since $\theta_k^{1/k} \leq e$ then (3.12) is implied by

$$\left(6\, e\, a_1/c_3^{1+1/\delta}\right)^k < e^{c_2 N/2},$$

which, in turn, is yield by

$$6\, e\, a_1 < \left(e^{c_2\, \mu_0\, /2}\, c_3\right)^{1+1/\delta}.$$

Since $6\, e\, a_1 \geq 1$, it is clear that the latter inequality will be satisfied once we choose $\mu_0 \geq 1$ so that, for example, $\exp(c_2\mu_0/2)\, c_3 > 6\, e\, a_1$. This can be trivially done with μ_0 depending on a_1, c_2 and c_3. $\Box$

Remark. Let us reiterate that the above theorem is valid for sections generated by n vectors in $\mathbb{R}^N$ with random ± 1 coordinates. Here $N = (1+\delta)n$ as above, and μ_0, c_3, $\tilde{c}_1$ and c_2' are absolute constants.

Let us finish with one more application of our results, which could be called a "lower M^*-estimate" for subspaces generated by columns of random matrices satisfying conditions *(M1)* and *(M2)*. In fact the estimate below follows immediately from the known results, by combining a "lower M^*-estimate", in the form proved in [Mi] or [PT], with Theorem 3.4; but the methods developed in this paper allow a direct and relatively shorter proof. We shall prove the result for matrices from $\mathrm{M}'(N, n, b, a_1, a_2)$. An analogous inequality for a general matrix satisfying *(M1)* and *(M2)* is quite obvious and is left for the interested reader.

Recall that for a symmetric convex body $K \subset \mathbb{R}^N$ we let

$$M^*(K) = \int_{S_{N-1}} \sup_{y \in K} |(x, y)|\, dx.$$

Theorem 3.5 *Let $\delta > 0$, $n > 1$ and $N = (1+\delta)n$, and let $a_1, b \geq 1$ and $a_2 > 0$. Let $\Gamma \in \mathrm{M}'(N, n, b, a_1, a_2)$ and let $K \subset \mathbb{R}^N$ be a symmetric convex body. There exist $\tilde{c}_1, c_2', C > 0$ and $c_3' > 1$ such that whenever $n \geq \tilde{c}_1^{1+1/\delta}$ then, with probability $\geq 1 - \exp(-c_2' N)$, we have*

$$\mathrm{diam}\,(K \cap \Gamma(\mathbb{R}^n)) \leq C\, a_1\, {c_3'}^{1+1/\delta} M^*(K). \tag{3.13}$$

Here c_3' and $\tilde{c}_1$ depend on b and a_1, while C and c_2' depend on b and a_2.

Proof: Recall that by Sudakov's inequality (cf. e.g., [P]), for every symmetric convex body $K \subset \mathbb{R}^N$ and every $\varepsilon > 0$ we have

$$\varepsilon \sqrt{\log N(K, \varepsilon B_2^N)} \leq C' M^*(K) \sqrt{N}, \tag{3.14}$$

where $C' > 0$ is an absolute constant.

Now, let $\tilde{c}_1, c_2 > 0$ and $0 < c_3 < 1$ be from Lemma 3.3, and set $c'_3 = 1/c_3$. Let

$$\rho = (2C'/\sqrt{c_2})\, a_1\, {c'_3}^{1+1/\delta}\, M^*(K),$$

and $\varepsilon = (2C'/\sqrt{c_2})\, M^*(K)$.

We shall now use (3.4) with $t_0 = 1/{c'_3}^{1+1/\delta}$, $\nu_0 = \exp(-c_2 N)$ and $\nu_1 = \exp(-a_2 N)$. Then, by Lemma 3.3, we get that whenever $n \geq \tilde{c}_1^{1+1/\delta}$, then

$$\mathrm{diam}\,(K \cap \Gamma(\mathbb{R}^n)) \leq (2C'/c_2)\, a_1\, {c'_3}^{1+1/\delta} M^*(K),$$

with probability $\geq 1 - N_\rho(\varepsilon)\exp(-c_2 N) - \exp(-a_2 N)$. Thus it is enough to notice that, by (3.14),

$$N_\rho(\varepsilon) \leq N(K, \varepsilon B_2^N) \leq e^{{C'}^2 M^*(K)^2 N/\varepsilon^2} \leq e^{c_2 N/4},$$

to get the lower estimate for the probability $\geq 1 - \exp(-3c_2N/4) - \exp(-a_2 N)$ $\geq 1 - \exp(-c'_2 N)$, for a suitable choice of $c'_2 > 0$ depending on b and a_2 only. $\square$

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