Euclidean embeddings in spaces of finite volume ratio via random matrices

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Abstract

Let $(\mathbb{R}^N, \|\cdot\|)$ be the space $\mathbb{R}^N$ equipped with a norm $\|\cdot\|$ whose unit ball has a bounded volume ratio with respect to the Euclidean unit ball. Let Γ be any random $N \times n$ matrix with $N > n$, whose entries are independent random variables satisfying some moment assumptions. We show that with high probability Γ is a good isomorphism from the n-dimensional Euclidean space $(\mathbb{R}^n, |\cdot|)$ onto its image in $(\mathbb{R}^N, \|\cdot\|)$, i. e. there exist $\alpha, \beta > 0$ such that for all $x \in \mathbb{R}^n$, $\alpha\sqrt{N}|x| \le \|\Gamma x\| \le \beta\sqrt{N}\,|x|$. This solves a conjecture of Schechtman on random embeddings of ℓ_2^n into ℓ_1^N.

1 Introduction

One of important steps in the study of Euclidean subspaces of finite-dimensional normed spaces was a result of Kashin ([Ka]) who proved that there exist subspaces of $\mathbb{R}^N$ of dimension proportional to N on which the ℓ_1 and ℓ_2 norms are equivalent. More precisely, for an arbitrary $\delta > 0$, let n and N be integers satisfying $N \ge n(1+\delta)$. Denote by $\|\cdot\|_1$ and $|\cdot|$ the ℓ_1 and the

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Euclidean norm on $\mathbb{R}^N$, respectively. Then there exists a subspace $E \subset \mathbb{R}^N$ of dimension n such that

$$\bar{a}(\delta)\sqrt{N}|x| \le \|x\|_1 \le \sqrt{N}|x| \qquad \text{for all } x \in E, \tag{1.1}$$

for a certain function $\bar{a}(\delta)$. In fact, "random" n-dimensional subspaces $E \subset \mathbb{R}^N$ (in sense of the Haar measure on the Grassman manifold $G_{N,n}$) satisfy condition (1.1). When $N = 2n$, this implies a well-known result of Kashin on the orthogonal decomposition of ℓ_1^N. Kashin's theorem was reproved by Szarek [Sz] by a different argument, which also worked in a more general case of spaces with so-called bounded volume ratio ([SzT], see also [Pi]).

Let $N \ge n$. In this paper we are interested in "random" sections of convex bodies in $\mathbb{R}^N$ given by n-dimensional subspaces of $\mathbb{R}^N$, spanned by the columns of rectangular $N \times n$ matrices Γ, whose entries are independent real-valued random variables on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

Observe that the Haar measure on the Grassman manifold $G_{N,n}$ is induced by an $N \times n$ matrix Γ whose entries are independent standard Gaussian random variables. Recently Schechtman studied in [S] an analogue of (1.1) for subspaces of ℓ_1^N spanned by the columns of matrices build from Bernoulli ± 1 variables. More precisely, he has shown that for a $2n \times n$ matrix A such that $A^* = \begin{bmatrix} \sqrt{n} I_n & B \end{bmatrix}$, where I_n is the identity $n \times n$ matrix, and B is an $n \times n$ matrix whose entries are independent Bernoulli ± 1 variables, the subspace spanned by the columns of A satisfies (1.1) with probability exponentially close to 1. He further showed ([S], Proposition 3) that for any $\delta > 0$ and $N \ge (1 + \delta)n$, there exists an $N \times n$ (non-random) matrix consisting of ± 1 entries only, whose columns span a subspace satisfying (1.1), and he conjectured that the result remains valid for "random" ± 1 matrices as well. In a related direction, it has been recently shown in [MiP] that the kernels of a random ± 1 matrix of size $n \times N$ also satisfy (1.1) with probability exponentially close to 1.

In this paper we consider a general class M of matrices Γ, defined in (2.1) and (2.2) below, which includes matrices with subgaussian entries (with uniform parameters), in particular matrices with standard Gaussian entries or Bernoulli ± 1 entries. We also consider a general class of centrally symmetric convex bodies in $\mathbb{R}^N$ with bounded volume ratio with respect to the Euclidean ball (see the definition in Section 2 below). We prove an analogue of (1.1) for normed spaces determined by such bodies and for subspaces spanned by the columns of matrices from our class. Moreover the result holds with probability exponentially (in N) close to 1. In particular this answers the question of Schechtman, for all δ, for spaces with bounded volume ratio, and sections spanned by the columns of matrices from M.

Our approach is a combination of probabilistic estimates in a Euclidean setting for individual vectors, and entropy arguments for general convex bodies. The former are based in a significant part on the methods developed in [LPRT]. However the estimates here are more delicate; in particular, Theorem 3.3 below is an extension of Theorem 3.1 from [LPRT]. The entropy arguments are new and seem to be of independent interest. The results of this paper were announced in [LPRTV]; however the outline of the argument given there lead to a weaker dependence of constants on δ.

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2 Preliminaries

By $|\cdot|$ and $\langle\cdot,\cdot\rangle$ we denote the canonical Euclidean norm and the canonical inner product on $\mathbb{R}^m$; and by B_2^m and S^{m-1}, the corresponding unit ball and the unit sphere, respectively. For $1 \le p \le \infty$, by $\|\cdot\|_p$ we denote the ℓ_p-norm, i.e., $\|a\|_p = \left(\sum_{i\ge 1}|a_i|^p\right)^{1/p}$ for $p<\infty$, and $\|a\|_\infty = \sup_{i\ge 1}|a_i|$. As usual, $\ell_p^m = (\mathbb{R}^m, \|\cdot\|_p)$, and the unit ball of ℓ_p^m is denoted by B_p^m.

For any Lebesgue measurable set $L \subset \mathbb{R}^m$, by $|L|$ we denote the volume of L. By a symmetric convex body we mean a centrally symmetric convex compact set with the non-empty interior.

Let $K \subset \mathbb{R}^m$ be a convex body whose interior contains the origin. For $x \in \mathbb{R}^m$ we denote by

$$\|x\|_K = \inf\{t \ge 0 \mid x \in tK\}$$

the Minkowski functional of K. We also set

$$V_K := \left(\frac{|K|}{|B_2^m|}\right)^{1/m}.$$

Whenever $B_2^m \subset K$ and V_K is bounded by a constant independent on the dimension, we say that K has bounded volume ratio with respect to the Euclidean ball. The prime example of such a body is $K = \sqrt{m} B_1^m$ which satisfies $B_2^m \subset K$ and

$$V_K = \left(\frac{2^m m^{m/2}}{m!} \; \frac{\Gamma(m/2+1)}{\pi^{m/2}} \right)^{1/m} \le \left(\frac{2e}{\pi} \right)^{1/2}.$$

(The $\Gamma(\cdot)$ above denotes the Gamma-function.)

The cardinality of a finite set A is denoted by $|A|$. This does not lead to a confusion with the volume, since the meaning of the notation will be always clear from the context.

Given a set $L \subset \mathbb{R}^m$, a convex body $K \subset \mathbb{R}^m$ and $\varepsilon > 0$, we say that a set A is an *ε-net* of L with respect to K if $A \subset L \subset \cup_{x \in A} (x + \varepsilon K)$. It is well known that if $K = L$ is a symmetric convex body (or if K is the boundary of a symmetric convex body L) then for every $\varepsilon > 0$ there exists an ε-net A of K with respect to L with cardinality $|A| \le (1 + 2/\varepsilon)^m$ (see e.g. [MiS], [Pi], [T]).

For a finite subset $\sigma \subset \{1, 2, \ldots, m\}$ we denote by P_σ the coordinate projection onto $\mathbb{R}^\sigma$. Sometimes we consider P_σ as an operator $\mathbb{R}^m \to \mathbb{R}^m$ and sometimes as an operator $\mathbb{R}^m \to \mathbb{R}^\sigma$.

Given a number a we denote the largest integer not exceeding a by $[a]$ and the smallest integer larger than or equal to a by $\lceil a \rceil$.

By g, g_i, $i \ge 1$, we denote independent $N(0,1)$ Gaussian random variables. By $\mathbb{P}(\cdot)$ we denote the probability of an event, and $\mathbb{E}$ denotes the expectation.

In this paper, we are interested in rectangular $N \times n$ matrices Γ, with $N \ge n$, whose entries are independent real-valued random variables on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We consider these matrices as operators acting from the Euclidean space ℓ_2^n to ℓ_2^N and we denote by $\|\Gamma\|$ the norm of Γ in $\mathrm{L}(\ell_2^n, \ell_2^N)$. If entries of Γ are independent $N(0,1)$ Gaussian variables we say that Γ is *a Gaussian random matrix.* If the entries of Γ are independent ± 1 Bernoulli random variables we say that Γ is *a ± 1 random matrix.*

For $\mu \ge 1$ and $a_1, a_2 > 0$, we let $\mathrm{M}(N, n, \mu, a_1, a_2)$ be the set of $N \times n$ matrices $\Gamma = (\xi_{ij})_{1 \le i \le N, 1 \le j \le n}$ whose entries are real-valued independent symmetric random variables ξ_{ij} on $(\Omega, \mathcal{A}, \mathbb{P})$ satisfying:

$$1 \le \|\xi_{ij}\|_{L^2} \quad \text{and} \quad \|\xi_{ij}\|_{L^3} \le \mu \text{ for every } \ 1 \le i \le N, \ 1 \le j \le n \tag{2.1}$$

$$\text{and } \ \mathbb{P}\left(\|\Gamma\| \ge a_1\sqrt{N}\right) \le e^{-a_2 N}. \tag{2.2}$$

Condition (2.2) on the operator norm is not easy to check in general, even for the case of independent identically distributed variables. However there is a natural class of random variables for which this condition can be shown to hold. Namely, a real-valued random variable ξ on $(\Omega, \mathcal{A}, \mathbb{P})$ is called *subgaussian* if there exists $b > 0$ such that for all $t > 0$ one has

$$\mathbb{E}\, e^{t\xi} \le e^{b^2 t^2/2}. \tag{2.3}$$

To emphasize the role of the parameter b we shall call a variable satisfying (2.3) b-subgaussian. If ξ is b-subgaussian, then it is classical to check by Chebyshev inequality and an easy optimization argument that

$$\mathbb{P}(\xi \ge u) \le \exp\left(-\frac{u^2}{2b^2}\right) \quad \text{for any } u \ge 0. \tag{2.4}$$

It can be also shown by a direct computation that (2.4) implies that ξ is $\bar{b}$-subgaussian, with $\bar{b} = 2b$.

The following basic fact, proved in [LPRT], makes subgaussian variables the main example in our context.

Fact 2.1 *Let $b > 0$, $a_2 > 0$ and $N \ge n \ge 1$. Let $\Gamma = (\xi_{ij})_{1 \le i \le N, 1 \le j \le n}$ be a matrix with independent b-subgaussian symmetric entries. Then*

$$\Gamma \in \mathrm{M}(N, n, b, a_1, a_2), \tag{2.5}$$

where $a_1 = 6\, b\sqrt{2(a_2 + 4)}$.

The notable examples of subgaussian variables are the standard Gaussian variables, the Bernoulli ± 1 variables, and variables uniformly distributed on the interval $[-1, 1]$. This makes our results applicable, in particular, for Gaussian random matrices and ± 1 random matrices.

Finally, we discuss how under some additional but still general assumptions (2.2) implies subgaussian-type estimates. If N and n are fixed and the entries of each column of a matrix Γ are independent identically distributed (i.i.d.) random variables then condition (2.2) is very close to the subgaussian estimates (2.3) and (2.4). More precisely, let ξ_1, ξ_2, ..., ξ_n be symmetric random variables and let $\Gamma = (\xi_{ij})_{i \le N, j \le n}$ be a matrix such that for every $j \le n$ the random variables ξ_{1j}, ξ_{2j}, ..., ξ_{Nj} are independent copies of ξ_j. Assume that Γ satisfies (2.2). For every $i \le N$ and $j \le n$ denote by

$$\bar{\xi_j} = \xi_j \cdot \mathbf{1}\left(|\xi_j| \le a_1\sqrt{N}\right) \quad \text{and} \quad \bar{\xi_{ij}} = \xi_{ij} \cdot \mathbf{1}\left(|\xi_{ij}| \le a_1\sqrt{N}\right)$$

the truncated random variables and let $\bar{\Gamma} = \left(\bar{\xi_{ij}}\right)_{i \le N, j \le n}$. Then the entries of each column j of $\bar{\Gamma}$ are independent copies of $\bar{\xi_j}$. Moreover, since by (2.2),

$$\mathbb{P}\left(|\xi_{ij}| \ge a_1\sqrt{N}\right) \le e^{-a_2 N},$$

we have

$$\mathbb{P}\left(\bar{\Gamma} = \Gamma\right) \ge 1 - N\, n\, e^{-a_2 N}.$$

We show that $\bar{\xi_j}$, $j \le n$, are subgaussian. Fix $j \le n$. Note that (2.2) applied to the standard basic vector e_j yields

$$\mathbb{P}\left(\sum_{i=1}^{N} (\bar{\xi_{ij}})^2 \ge a_1^2 N\right) \le \mathbb{P}\left(\sum_{i=1}^{N} \xi_{ij}^2 \ge a_1^2 N\right) = \mathbb{P}\left(|\Gamma e_j| \ge a_1\sqrt{N}\right) \le e^{-a_2 N}. \tag{2.6}$$

This inequality implies that for every $k \le N$ one has

$$\mathbb{P}\left((\bar{\xi_{ij}})^2 \ge a_1^2 N/k \ \text{ for all } \ i \le k\right) \le e^{-a_2 N}.$$

Since $\bar{\xi_{ij}}$, $i \le N$, are i.i.d., we obtain

$$\mathbb{P}\left((\bar{\xi_j})^2 \ge a_1^2 N/k\right) = \left(\prod_{i=1}^{k} \mathbb{P}\left((\bar{\xi_{ij}})^2 \ge a_1^2 N/k\right)\right)^{1/k} \le e^{-a_2 N/k}.$$

Now let $a_1 \le t \le a_1\sqrt{N}$. Choose k such that $a_1^2 N/(k+1) \le t^2 \le a_1^2 N/k$. Then

$$\mathbb{P}(|\bar{\xi_j}| > t) \le \exp\left(-a_2 \frac{N}{k+1}\right) \le \exp\left(-\frac{a_2}{2a_1^2}\, t^2\right).$$

Since ξ_j is symmetric and $\mathbb{P}(|\bar{\xi_j}| > a_1\sqrt{N}) = 0$, standard calculations show that ξ_j is b-subgaussian, where b depends on a_1 and a_2 only.

We would like to note that the truncation of the ξ_j's is necessary since condition (2.2) does not provide any information about the events of probability less than $\exp(-a_2 N)$. On the other hand, by considering a sequence of conditions rather than just one condition as above, we may indeed conclude that the entries of a matrix are subgaussian, even without the truncation. More precisely, let ξ be a random variable and let $\{\xi_i\}_i$ be a sequence of non-necessarily independent copies of ξ satisfying for every $m \geq 1$ the estimate

$$\mathbb{P}\left(\sum_{i=1}^{m} \xi_i^2 \geq a_1^2 m\right) \leq e^{-a_2 m} \tag{2.7}$$

(compare with (2.6)). Then $\mathbb{P}\left(\xi \geq a_1\sqrt{m}\right) \leq e^{-a_2 m}$ for every m and, by a similar calculation as above, it can be shown that ξ is subgaussian. In other words if $\Gamma = (\xi_{ij})_{i\geq 1, j\geq 1}$ is an infinite matrix such that for each fixed j the sequence $(\xi_{ij})_i$ is a sequence of identically (non-necessarily independent) distributed random variables satisfying (2.7) for every m, then for every $N \geq 1$ and every $n \leq N$ the matrix $\Gamma_{N,n} = (\xi_{ij})_{i\leq N, j\leq n}$ has subgaussian entries. In particular, if the entries are independent, $\Gamma_{N,n}$ satisfies (2.2).

In the conclusion of this section let us mention that in the case of identically distributed independent entries, the boundedness of the norms of a sequence of (properly normalized) matrices implies the finiteness of the fourth moment of the entries (see [BSY]).

3 Main results

Our main theorem says.

Theorem 3.1 *Let $\delta > 0$, let $n \geq 1$ and $N = (1+\delta)n$. Let Γ be an $N \times n$ matrix from $\mathrm{M}(N, n, \mu, a_1, a_2)$, for some $\mu \geq 1$, $a_1, a_2 > 0$. Let $K \subset \mathbb{R}^N$ be a symmetric convex body such that $B_2^N \subset K$. There exist constants $C_1 \geq 1$ and $c_1 \geq 1$ depending on a_1 and μ only, such that whenever $n \geq (C_1 V_K)^{1+1/\delta}$, we have*

$$\mathbb{P}\big(\|\Gamma x\|_K \geq (c_1 V_K)^{-(c_1+1/\delta)}\sqrt{N}\,|x| \ \text{ for all } x \in \mathbb{R}^n\big) \geq 1 - \exp(-c_2 N),$$

where $c_2 > 0$ depends on μ and a_2.

Remark 1. Our proof below gives that the desired probability can be made less than $\exp(-N)+\exp(-cN/\mu^6)+\exp(-a_2N)$, where c is an absolute positive constant.
Remark 2. Since for every convex body $K\subset\mathbb{R}^N$ with 0 in its interior we have $|K-K|\le 2^N|K|$ and $\|\cdot\|_K\ge\|\cdot\|_{K-K}$, Theorem 3.1 holds for every convex body $K\in\mathbb{R}^N$ with 0 in its interior.

We have an immediate corollary, the second part of which is an analogue of (1.1) and solves a substantial generalization of the aforementioned question of Schechtman.

Corollary 3.2 *Under the assumptions of Theorem 3.1, the subspace E spanned by the n columns of the matrix Γ satisfies, with $\alpha := (c_1V_K)^{-(c_1+1/\delta)}$, and with probability $\ge 1-e^{-c_2N}$,*

$$(1/a_1\sqrt{N})\Gamma(B_2^n)\subset B_2^N\cap E\subset K\cap E\subset(1/\alpha\sqrt{N})\Gamma(B_2^n)\subset(a_1/\alpha)B_2^N\cap E$$

and

$$\sqrt{N}\alpha|x|\le\|\Gamma x\|_K\le a_1\sqrt{N}\,|x|\quad \text{for all } x\in\mathbb{R}^n.$$

As we mentioned in introduction the proof of Theorem 3.1 combines probability results, parallel to the proof of the main result from [LPRT], and new covering estimates (see Proposition 3.10 below). In particular, a slight modification of the argument from [LPRT] is sufficient to show the following extension of the main result from that paper.

Theorem 3.3 *Let $\delta>0$, let $n\ge 1$ and $N=(1+\delta)n$. Let Γ be an $N\times n$ matrix from $\mathrm{M}(N,n,\mu,a_1,a_2)$, for some $\mu\ge 1$, $a_1,a_2>0$. There exist positive constants C_1 and c_1 depending on a_1 and μ only, such that whenever $n\ge C_1^{(1+\delta)/\delta}$, then, for every fixed $w\in\mathbb{R}^N$, we have*

$$\mathbb{P}\left(\exists x\in S^{n-1}\ s.t.\ \Gamma x\in w+c_1^{(1+\delta)/\delta}\ \sqrt{N}B_2^N\right)\le\exp(-c_2N),$$

where $c_2>0$ depends on μ and a_2.

Remark. It is noteworthy that, as can be seen from the proof below, the case when $\delta\ge\delta_0$, where $\delta_0>0$ is a certain (large) absolute constant, is much simpler than the case of a general (small) δ. Indeed, this former case follows directly from Proposition 3.6, without use of Proposition 3.4.

The proofs of both theorems are based on two key propositions. The first result will be used to estimate a single coordinate (hence $\|\cdot\|_\infty$), of the vector Γx, for a fixed $x \in \mathbb{R}^n$. We state it here in a more general form, as we believe it is of an independent interest.

Recall that for any subset $\sigma \subset \{1, \dots, n\}$, P_σ denotes the coordinate projection in $\mathbb{R}^n$.

Proposition 3.4 *Let $(\xi_i)_{i=1}^n$ be a sequence of symmetric independent random variables with $1 \le \|\xi_i\|_{L_2} \le \|\xi_i\|_{L_3} \le \mu$ for all $i = 1, \dots, n$. Then for any $x = (x_i) \in \mathbb{R}^n$, $\sigma \subset \{1, \dots, n\}$ we have, for all $s \in \mathbb{R}$ and $t > 0$,*

$$\mathbb{P}\Big(\big|\sum_{i=1}^n \xi_i x_i - s\big| < t\Big) \le \sqrt{2/\pi}\frac{t}{|P_\sigma x|} + c\Big(\frac{\|P_\sigma x\|_3}{|P_\sigma x|}\mu\Big)^3,$$

where $c > 0$ is a universal constant.

This proposition depends on the well-known Berry-Esséen theorem (cf., e.g., [St], Section 2.1).

Lemma 3.5 *Let $(\zeta_i)_{i=1}^n$ be a sequence of symmetric independent random variables with finite third moments, and let $A^2 := \sum_{i=1}^n \mathbb{E}|\zeta_i|^2$. Then for every $\tau \in \mathbb{R}$ one has*

$$\Big|\mathbb{P}\Big(\sum_{i=1}^n \zeta_i < \tau A\Big) - \mathbb{P}\left(g < \tau\right)\Big| \le (c/A^3)\sum_{i=1}^n \mathbb{E}|\zeta_i|^3,$$

where g is a Gaussian random variable with $N(0,1)$ distribution and $c \ge 1$ is a universal constant.

Proof of Proposition 3.4: First we show a stronger estimate for $\sigma = \{1, \dots, n\}$. Namely, for any $a < b$,

$$\mathbb{P}\Big(\sum_{i=1}^n \xi_i x_i \in [a, b)\Big) \le \sqrt{1/2\pi}\,\frac{b-a}{|x|} + c\Big(\frac{\|x\|_3}{|x|}\mu\Big)^3, \qquad (3.1)$$

where $c > 0$ is a universal constant.

Indeed, let $\zeta_i = \xi_i x_i$. Then $A^2 := \sum_i \mathbb{E}\zeta_i^2 = \sum_i x_i^2 \mathbb{E}\xi_i^2 \geq |x|^2$ and $\mathbb{E}\sum_i |\zeta_i|^3 \leq \mu^3 \|x\|_3^3$. By Lemma 3.5 we get

$$\begin{aligned}
\mathbb{P}\Big(a \leq \sum_{i=1}^n \zeta_i < b\Big) &\leq \mathbb{P}\big(a/A \leq g < b/A\big) + c\Big(\frac{\|x\|_3}{A}\mu\Big)^3 \\
&\leq \frac{b-a}{A\sqrt{2\pi}} + c\Big(\frac{\|x\|_3}{A}\mu\Big)^3 \\
&\leq \sqrt{1/2\pi}\,\frac{b-a}{|x|} + c\Big(\frac{\|x\|_3}{|x|}\,\mu\Big)^3,
\end{aligned}$$

as required.

Now, if σ is arbitrary, denote the sequence $(\xi_i)_{i\in\sigma}$ by (ξ_i') and the sequence $(\xi_i)_{i\notin\sigma}$ by (ξ_i''), and by $\mathbb{P}'$ (resp., $\mathbb{P}''$) and $\mathbb{E}'$ (resp., $\mathbb{E}''$) the corresponding probabilities and expectations. The independence and Fubini theorem imply

$$\begin{aligned}
\mathbb{P}\Big(\Big|\sum_{i=1}^n \xi_i x_i - s\Big| < t\Big) &= \mathbb{P}\Big(s - t - \sum_{i=1}^n \xi_i'' x_i < \sum_{i=1}^n \xi_i' x_i < s + t - \sum_{i=1}^n \xi_i'' x_i\Big) \\
&= \mathbb{E}''\mathbb{P}'\Big(s - t - \sum_{i=1}^n \xi_i'' x_i < \sum_{i=1}^n \xi_i' x_i < s + t - \sum_{i=1}^n \xi_i'' x_i\Big) \\
&\leq \sqrt{1/2\pi}\,\frac{2t}{|P_\sigma x|} + c\Big(\frac{\|P_\sigma x\|_3}{|P_\sigma x|}\mu\Big)^3.
\end{aligned}$$

The latter inequality follows from (3.1), the fact that the vector appearing in the sum $\sum_i \xi_i' x_i$ is exactly $P_\sigma x$, and by the independence of the ensembles $(\xi_i)_{i\in\sigma}$ and $(\xi_i)_{i\notin\sigma}$. □

Our second proposition is a general estimate for the norm $|\Gamma x|$ for a fixed vector x.

Proposition 3.6 *Let $1 \leq n < N$ be positive integers. Let Γ be an $N \times n$ random matrix from $\mathrm{M}(N, n, \mu, a_1, a_2)$, for some $\mu \geq 1$ and $a_1, a_2 > 0$. Then for every $x \in \mathbb{R}^n$ and every $w \in \mathbb{R}^N$ we have*

$$\mathbb{P}\left(|\Gamma x - w| \leq c'\mu^{-3}\sqrt{N}\,|x|\right) \leq \exp\left(-c'' N/\mu^6\right),$$

where $0 < c', c'' < 1$ are absolute constants.

The proof of this proposition will be using the following simple estimate which is a general form of the Paley-Zygmund inequality (see e.g. [LPRT] for the formulation used here).

Lemma 3.7 *Let $p \in (1, \infty)$, $q = p/(p-1)$. Let $f \geq 0$ be a random variable with $\mathbb{E} f^{2p} < \infty$. Then for every $0 \leq \lambda \leq \sqrt{\mathbb{E} f^2}$ we have*

$$\mathbb{P}(f > \lambda) \geq \frac{(\mathbb{E} f^2 - \lambda^2)^q}{(\mathbb{E} f^{2p})^{q/p}}.$$

Corollary 3.8 *Let $\mu \geq 1$ and $(\xi_i)_{i \geq 1}$ be a sequence of independent symmetric random variables such that $1 \leq \mathbb{E}|\xi_i|^2 \leq \mathbb{E}|\xi_i|^3 \leq \mu^3$ for every $i \geq 1$. Let $s \in \mathbb{R}$, $x = (x_i)_{i \geq 1} \in \ell_2$ be such that $|x| = 1$ and $f = |\sum_{i \geq 1} x_i \xi_i - s|$. Then for every $0 \leq \lambda \leq 1$ one has*

$$\mathbb{P}(f > \lambda) \geq \frac{(1-\lambda^2)^3}{32 \mu^6}.$$

Proof: Set $h = \sum_{i \geq 1} x_i \xi_i$. By the symmetry of the ξ_i's and Khinchine's inequality ([H]),

$$\mathbb{E}|h|^3 = \mathbb{E}_\xi \mathbb{E}_\varepsilon \left| \sum_{i \geq 1} \varepsilon_i \xi_i x_i \right|^3 \leq \sqrt{8}\ \mathbb{E}_\xi \left(\sum_{i \geq 1} \xi_i^2 x_i^2 \right)^{3/2},$$

where ε_i's are independent Bernoulli ± 1 random variables. (In the inequality above we used the estimate for the Khinchine's constant $B_3 = \sqrt{2}\pi^{-1/6} \leq \sqrt{2}$, while the standard proof gives $B_3 \leq 2$.) Consider the function $\varphi(s)$ defined on the set

$$E := \left\{ s = (s_i)_{i \geq 1} \in \ell_1 \mid s_i \geq 0 \text{ for every } i \text{ and } \sum_{i \geq 1} s_i = 1 \right\}$$

by

$$\varphi(s) = \mathbb{E}_\xi \left(\sum_{i \geq 1} \xi_i^2 s_i \right)^{3/2}.$$

Clearly φ is convex, since a power larger than 1 of a linear function is convex. Thus to estimate the supremum of φ it is enough to estimate the supremum of values $\varphi(e_i)$ for the standard unit vectors $e_i \in \ell_1$. Therefore

$$\sup_E \varphi(s) = \sup_{i \geq 1} \varphi(e_i) = \sup_{i \geq 1} \mathbb{E}_\xi \left(\xi_i^2 \right)^{3/2} \leq \mu^3,$$

which implies

$$\mathbb{E}|h|^3 \le \sqrt{8}\mu^3.$$

Since the ξ_i's are symmetric we have that h is symmetric. Using this and a simple inequality $|a|^3+|b|^3 \le (a^2+b^2)^{3/2} \le \sqrt{2}(|a|^3+|b|^3)$ we obtain that

$$\begin{aligned}\mathbb{E}f^3 &= \mathbb{E}|h-s|^3 = (1/2)\mathbb{E}\left(|h-s|^3+|-h-s|^3\right)\\ &\le (1/2)\mathbb{E}\left(|h-s|^2+|h+s|^2\right)^{3/2} = (1/2)\mathbb{E}\left(2|h|^2+2s^2\right)^{3/2}\\ &\le 2(\mathbb{E}|h|^3+|s|^3) \le 4\sqrt{2}\mu^3+2|s|^3.\end{aligned}$$

Next, by our normalization,

$$\mathbb{E}h^2 = \mathbb{E}\sum_{i\ge 1}\xi_i^2|x_i|^2 \ge 1.$$

Using the symmetry of h again we observe

$$\mathbb{E}f^2 = \mathbb{E}|h-s|^2 = (1/2)\mathbb{E}\left(|h-s|^2+|-h-s|^2\right) = \mathbb{E}\left(|h|^2+s^2\right) \ge 1+s^2.$$

Applying Lemma 3.7 with $p=3/2$ we obtain

$$\mathbb{P}\left(f>\lambda\right) \ge \frac{(1+s^2-\lambda^2)^3}{(4\sqrt{2}\mu^3+2|s|^3)^2}.$$

The right hand side is minimal when $s=0$ and this implies the desired result. $\square$

Proof of Proposition 3.6 Let $x=(x_i)_i\in\mathbb{R}^n$ with $|x|=1$. Let $\Gamma=(\xi_{ji})_{j\le N,i\le n}$ where ξ_{ji} are independent symmetric random variables with $1\le\|\xi_{ji}\|_{L_2}\le\|\xi_{ji}\|_{L_3}\le\mu$, for every $j\le N$ and every $i\le n$. Let $f_j=|\sum_{i=1}^n\xi_{ji}x_i-w_j|$. Note that $f_1,\dots,f_N$ are independent. For any $t,\tau>0$ we have

$$\begin{aligned}\mathbb{P}\left(|\Gamma x-w|^2\le t^2N\right) &= \mathbb{P}\left(\sum_{j=1}^N f_j^2\le t^2N\right) = \mathbb{P}\left(N-\frac{1}{t^2}\sum_{j=1}^N f_j^2\ge 0\right)\\ &\le \mathbb{E}\exp\left(\tau N-\frac{\tau}{t^2}\sum_{j=1}^N f_j^2\right) = e^{\tau N}\prod_{j=1}^N\mathbb{E}\exp\left(-\tau f_j^2/t^2\right).\end{aligned}$$

To estimate the latter expectations first observe that by Corollary 3.8, for every $\lambda \in (0,1)$ and for every j

$$\mathbb{P}\left(f_j > \lambda\right) \geq \frac{(1-\lambda^2)^3}{32\mu^6} =: \beta.$$

Therefore, by the distribution function formula,

$$\begin{aligned}
\mathbb{E}\exp\left(-\tau f_j^2/t^2\right) &= \int_0^\infty \mathbb{P}\left(\exp\left(-\tau f_j^2/t^2\right) > s\right) ds \\
&= \int_0^1 \mathbb{P}\left(1/s > e^{\tau f_j^2/t^2}\right) ds \\
&\leq \int_0^{e^{-\tau\lambda^2/t^2}} ds + \int_{e^{-\tau\lambda^2/t^2}}^1 (1-\beta) ds \\
&= e^{-\tau\lambda^2/t^2} + (1-\beta)\left(1 - e^{-\tau\lambda^2/t^2}\right) \\
&= 1 - \beta\left(1 - e^{-\tau\lambda^2/t^2}\right).
\end{aligned}$$

Set $\tau = \alpha t^2/\lambda^2$, for some $\alpha > 0$. Then for any $t > 0$ we get, for arbitrary $\alpha > 0$ and $\lambda \in (0,1)$,

$$\mathbb{P}\left(|\Gamma x - w|^2 \leq t^2 N\right) \leq \left(e^{\alpha t^2/\lambda^2}\left(1 - \beta(1-e^{-\alpha})\right)\right)^N. \tag{3.2}$$

For example, letting $\lambda = 1/2$ we get $\beta = 27/(2^{11}\mu^6)$, and using $1 - s < e^{-s}$ for $s > 0$, the left hand side expression in (3.2) is less than

$$\exp\left(\left(4\alpha t^2 - \beta(1-e^{-\alpha})\right) N\right).$$

Thus letting $\alpha = \ln 2$ and $t = \sqrt{\beta}/4$ we conclude the required estimates with $c' = (27/2^{15})^{1/2}$ and $c'' = 27/2^{13}$. □

Now we prove a lemma which will imply both theorems. First we introduce more notations. Fix a, b to be defined later (they will depend on μ, a_1, a_2 and satisfy $0 < 2a \leq b \leq 1/4$). Given $x \in \mathbb{R}^n$ let $\sigma(x,a) := \{i : |x_i| \leq a\}$. We set

$$\begin{aligned}
S(a,b) &= S^{n-1} \cap \left\{x \ : \ |P_{\sigma(x,a)} x| \leq b\right\}, \\
S'(a,b) &= S^{n-1} \cap \left\{x \ : \ |P_{\sigma(x,a)} x| > b\right\},
\end{aligned}$$

and, given $w \in \mathbb{R}^N$ and $t > 0$, we denote

$$\Omega(w,t,a,b) = \overline{\Omega} \cap \Big(\exists x \in S(a,b) \text{ s.t. } |\Gamma x - w| \le t\sqrt{N}\Big), \tag{3.3}$$

$$\Omega'(w,t,a,b) = \overline{\Omega} \cap \Big(\exists x \in S'(a,b) \text{ s.t. } |\Gamma x - w| \le t\sqrt{N}\Big), \tag{3.4}$$

where $\overline{\Omega} = \Big\{\omega : \|\Gamma\| \le a_1\sqrt{N}\Big\}$.

We shall estimate the probabilities of these sets separately. In both cases the idea of the proof is the same. We shall estimate the probability that $|\Gamma x - w| \le t\sqrt{N}$ for a single vector x and then use the ε-net argument and approximation. However, the balance between the probabilistic estimate and the cardinality of an ε-net will be different in each case. If $x \in S'(a,b)$, we have a good control of the ℓ_∞-norm of the vector $P_{\sigma(x,a)}x$, which allows us to apply the powerful estimate of Proposition 3.4. In this case the standard estimate $(3/\varepsilon)^n$ for the cardinality of an ε-net on the sphere S^{n-1} will be sufficient. In the case when $x \in S(a,b)$, to bound the probability for a fixed x, we shall use a weaker, but more general estimate from Proposition 3.6. However, since in this case $|P_{\sigma(x,a)}x| \le b$, a vector x can be approximated by another vector having a small support. This observation yields a much better bound for the cardinality of an ε-net for $S(a,b)$.

Lemma 3.9 *Let $0 < \delta < 1$, $n \ge 1$ and $N = (1+\delta)n$. Let Γ be an $N \times n$ matrix from $\mathrm{M}(N,n,\mu,a_1,a_2)$, for some $\mu \ge 1$, $a_1, a_2 > 0$. Let $w \in \mathbb{R}^N$. There are absolute positive constants c_1, c_2, c_3 and a positive constant $\bar{c}$ depending only on a_1 and μ such that*

(i) for every $V \ge 1$, $b \le 1/4$, and a, t satisfying

$$aa_1 = t \le \min\left\{\frac{a_1 b}{2}\ ,\ \frac{b}{\bar{c}V}\left(\frac{b}{3\bar{c}a_1V}\right)^{1/\delta}\right\}$$

one has

$$\mathbb{P}\left(\Omega'(w,t,a,b)\right) \le V^{-N};$$

(ii) for $b = \min\{1/4,\ c_1/(a_1\mu^3)\}$, $t = a_1 b/2$ and every a satisfying

$$c_2\mu^3\sqrt{\ln(c_2\mu^6/b)}/\sqrt{n} \le a \le b/2$$

(assuming n is large enough) one has

$$\mathbb{P}\left(\Omega(w,t,a,b)\right) \le \exp\left(-c_3N/\mu^6\right).$$

Proof: *Case I: Probability of* $\Omega'(w,t,a,b)$. Let $\mathcal{N} \subset S^{n-1}$ be an ε-net in S^{n-1} of cardinality $|\mathcal{N}| \le (3/\varepsilon)^n$. Setting $\varepsilon := a = t/a_1 \le b/2$ a standard approximation argument shows that if there exists $x \in S^{n-1}$ such that $|\Gamma x - w| \le t\sqrt{N}$ and $|P_{\sigma(x,a)}x| > b$ then there exist $v \in \mathcal{N}$ and $\sigma \subset \{1,\dots,n\}$ such that

$$|\Gamma v - w| \le (t+\varepsilon a_1)\sqrt{N} = 2t\sqrt{N}, \quad \|P_\sigma v\|_\infty \le a+\varepsilon = 2a, \quad |P_\sigma v| \ge b-\varepsilon \ge b/2.$$

Denote by $\mathcal{A}$ the set of all $v \in \mathcal{N}$ for which there exists $\sigma = \sigma(v) \subset \{1,\dots,n\}$ such that

$$\|P_\sigma v\|_\infty \le 2a, \qquad |P_\sigma v| \ge b/2.$$

Then $|\mathcal{A}| \le |\mathcal{N}| \le (3/\varepsilon)^n$ and

$$\mathbb{P}\left(\Omega'(w,t,a,b)\right) \le \mathbb{P}\left(\exists v \in \mathcal{A} \,:\, |\Gamma v - w| \le 2t\sqrt{N}\right). \tag{3.5}$$

Now, fix $v = (v_i)_i \in \mathcal{A}$. For every $j = 1,\dots,N$, set

$$f_j(\lambda) = \mathbb{P}\Big(\Big|\sum_{i=1}^n \xi_{ji} v_i - w_j\Big| < \lambda\Big),$$

and let $f(\lambda) = \sup_j f_j(\lambda)$. Since $\|\cdot\|_3^3 \le \|\cdot\|_\infty |\cdot|^2$, by Proposition 3.4 we get

$$\begin{aligned} f(\lambda) &\le c\left(\lambda + \|P_\sigma v\|_\infty \mu^3\right)/|P_\sigma v| \\ &\le 2c\left(\lambda + 2a\mu^3\right)/b \le (4c/b)\max\left\{\lambda, 2a\mu^3\right\}, \end{aligned} \tag{3.6}$$

where $\sigma = \sigma(v)$ and $c \ge \sqrt{2/\pi}$ is an absolute constant.

Now we have

$$\begin{aligned} \mathbb{P}\Big(|\Gamma v - w|^2 \le 4t^2 N\Big) &= \mathbb{P}\Big(\sum_{j=1}^N |\sum_{i=1}^n \xi_{ji} v_i - w_j|^2 \le 4t^2 N\Big) \\ &= \mathbb{P}\Big(N - \sum_{j=1}^N |\sum_{i=1}^n \xi_{ji} v_i - w_j|^2/4t^2 \ge 0\Big) \\ &\le \mathbb{E}\exp\Big(N - \sum_{j=1}^N |\sum_{i=1}^n \xi_{ji} v_i - w_j|^2/4t^2\Big) \\ &= e^N \prod_{j=1}^N \mathbb{E}\exp\Big(-|\sum_{i=1}^n \xi_{ji} v_i - w_j|^2/4t^2\Big). \end{aligned}$$

We estimate the expectations by passing to the integral formula. Denote $A := \sqrt{2} a \mu^3 / t$. Then

$$\begin{aligned}
\mathbb{E} \exp\left(-|\sum_{i=1}^n \xi_{ji} v_i - w_j|^2/4t^2\right) &= \int_0^1 \mathbb{P}\Big(\exp\Big(-|\sum_{i=1}^n \xi_{ji} v_i - w_j|^2/4t^2\Big) > s\Big) ds \\
&= \int_0^\infty u e^{-u^2/2}\ \mathbb{P}\Big(|\sum_{i=1}^n \xi_{ji} v_i - w_j| < \sqrt{2} t u\Big) du \\
&= \int_0^\infty u e^{-u^2/2} f_j(\sqrt{2} t u) du \\
&\le (4c/b) \left(2 \int_0^A u a \mu^3\, du + \int_A^\infty \sqrt{2} t u^2 e^{-u^2/2}\, du\right) \\
&\le (4c/b) \left(a \mu^3 A^2 + t\sqrt{\pi}\right) \\
&= (4c/b) \left(2a^3 \mu^9/t^2 + t\sqrt{\pi}\right) \\
&= (4ct/b) \left(2\mu^9/a_1^3 + \sqrt{\pi}\right) = \bar{c} t/b,
\end{aligned}$$

where $\bar{c} := 4c\,(2\mu^9/a_1^3 + \sqrt{\pi})$. So

$$\mathbb{P}\Big(|\Gamma v - w|^2 \le 4t^2 N\Big) \le (\bar{c} e\, t/b)^N.$$

Finally, since $\varepsilon = a = t/a_1$, we get by (3.5),

$$\mathbb{P}\big(\Omega'(w,t,a,b)\big) \le |\mathcal{A}|\ (\bar{c} e\, t/b)^N \le \big(3a_1/t\big)^n (\bar{c} e\, t/b)^N \le V^{-N}, \tag{3.7}$$

for any t satisfying

$$t \le \frac{b}{e\bar{c}V} \left(\frac{b}{3e\bar{c}a_1 V}\right)^{1/\delta} \tag{3.8}$$

Case II: Probability of $\Omega(w,t,a,b)$. Given $x \in S^{n-1}$ recall that $\sigma(x,a) = \{i : |x_i| \le a\}$, and set $\sigma'(x,a) = \{1,\dots,n\} \setminus \sigma(x,a)$. By the definition of $\sigma(x,a)$, clearly, $|\sigma'(x,a)| \le [1/a^2] =: m$. Let $y = P_{\sigma'(x,a)} x$. If now x is a vector appearing in the definition (3.3) of $\Omega(w,t,a,b)$ then $|\Gamma y - w| \le (t + a_1 b)\sqrt{N}$, $|y| \ge (1-b^2)^{1/2} \ge 1 - b$ and $|\operatorname{supp}(y)| \le m$.

Of course we want $m \le n$, which is satisfied since $a \ge 1/\sqrt{n}$.

Let $\varepsilon = b$ and let $\mathcal{N} \subset B_2^n$ such that for every y' with $|y'| \le 1$ and the support $\le m$ there exists $v \in \mathcal{N}$ such that $|y' - v| \le \varepsilon$. We can chose $\mathcal{N}$

with cardinality $|\mathcal{N}| \leq \binom{n}{m}(3/\varepsilon)^m \leq (e\,n/m)^m(3/\varepsilon)^m$. Thus choosing v for y as above we get $v \in \mathcal{N}$ such that $|v| \geq |y| - \varepsilon \geq 1 - 2b \geq 1/2$ and

$$|\Gamma v - w| \leq (t + 2\,a_1 b)\sqrt{N} \leq (5/2)a_1 b\sqrt{N} \leq 5a_1 b\sqrt{N}|v|.$$

(We used the fact that $t = a_1 b/2$, by our conditions.) Thus, by Proposition 3.6, we get that if

$$b \leq c'/(5a_1\mu^3) \tag{3.9}$$

then

$$\mathbb{P}\big(\Omega(w,t,a,b)\big) \leq (e\,n/m)^m(3/b)^m \exp\left(-c''N/\mu^6\right) \leq \exp\left(-c''N/(2\mu^6)\right) \tag{3.10}$$

for m satisfying

$$m \ln\left(\frac{3en}{bm}\right) \leq c''N/(2\mu^6). \tag{3.11}$$

Since $m = [1/a^2] \leq n$, the last inequality holds if

$$(1/a^2)\ln\left(\frac{3ena^2}{b}\right) \leq c''n/(2\mu^6). \tag{3.12}$$

To satisfy the latter inequality it is enough to take a such that

$$1/a^2 \leq \frac{c''n}{4\mu^6 \ln\left(6e\mu^6/\left(c''b\right)\right)}. \tag{3.13}$$

This proves the lemma. $\square$

Theorem 3.3 immediately follows from Lemma 3.9.

Proof of Theorem 3.3: Let c_1, c_2, c_3, $\bar{c}$ be constants from Lemma 3.9. Let $V = e$, $b = \min\{1/4,\ c_1/(a_1\mu^3)\}$, $t_0 = a_1 b/2$, $t = \min\left\{t_0\ ,\ \frac{b}{\bar{c}V}\left(\frac{b}{3\bar{c}a_1 V}\right)^{1/\delta}\right\}$, $a = t/a_1$. Note that $\Omega(w,t,a,b) \subset \Omega(w,t_0,a,b)$. Therefore, by Lemma 3.9 (condition on a will be satisfied by an appropriate choice of C_1 in the conditions of Theorem 3.3) we obtain

$$\mathbb{P}\left(\Omega'(w,t,a,b)\right) \leq e^{-N} \quad \text{and} \quad \mathbb{P}\left(\Omega(w,t,a,b)\right) \leq \exp\left(-c_3 N/\mu^6\right).$$

To obtain Theorem 3.3 observe that we are interested in the set

$$\Big(\exists\, x \in S^{n-1} \text{ s.t. } |\Gamma x - w| \leq t\sqrt{N}\Big),$$

which is the union of $\Omega'(w,t,a,b)$, $\Omega(w,t,a,b)$ and complement of $\overline{\Omega}$. Moreover, by the definition of the class $\mathrm{M}(N,n,\mu,a_1,a_2)$ we also have that $\mathbb{P}(\overline{\Omega}) \geq 1-\exp(-a_2 N)$. Putting the three estimates together

$$\mathbb{P}\Big(\exists\, x \in S^{n-1} \text{ s.t. } |\Gamma x - w| \leq t\sqrt{N}\Big) \leq e^{-N} + e^{-c_3 N/\mu^6} + e^{-a_2 N},$$

which concludes the proof. $\square$

To prove Theorem 3.1 we need the following proposition.

Proposition 3.10 *There exists an absolute constant $c > 0$ such that for every $0 < r \leq 1/e$, every symmetric convex body $K \subset \mathbb{R}^N$ satisfying $B_2^N \subset K$, and every $0 < \eta \leq \ln(4\pi V_K)/\ln(1/r)$ one has*

$$N\left(\alpha K \cap B_2^N, r B_2^N\right) \leq 2^{\eta N} \quad \textit{for} \quad \alpha = (4\pi V_K)^{(c/\eta)\ln r}.$$

Proof: Set $L := \alpha K \cap B_2^N$, and $A = 4\pi V_K$. By Szarek's volume ratio theorem, for every $1 \leq k \leq n$ there exists a subspace $E \subset \mathbb{R}^N$ with codim $E = k$ such that $L \cap E \subset \min(1, \alpha A^{N/k}) B_2^N \cap E$. It is now convenient to use some terminology of so-called s-numbers of operators. Let $(\mathbb{R}^N, W)$ be a Banach space equipped with the norm defined by a symmetric convex body W. For an operator $u : (\mathbb{R}^N, W) \to \ell_2^N$ and any j, the j'th Gelfand number is defined by $c_j(u) = \inf\{\|u|_E\| : E \subset \mathbb{R}^N, \text{codim } E < j\}$, and the j'th entropy number is defined by $e_j(u) = \inf\{\varepsilon : N(u(W), \varepsilon B_2^N) \leq 2^{j-1}\}$. In particular, letting u to be the formal identity operator from $(\mathbb{R}^N, L)$ to ℓ_2^N, we have $c_{k+1}(u) \leq \min(1, \alpha A^{n/k})$.

Set $\beta = \ln(1/r) \geq 1$, and $m = [\eta N]$. By Carl's theorem ([C], cf., [Pi] Th. 5.2) one has

$$m^\beta e_m(u) \leq \rho_\beta \sup_{k \leq m} k^\beta c_{k+1}(u).$$

Moreover, following precisely the proof of Carl's theorem, it can be observed that for $\beta \geq 1$ one can take $\rho_\beta \leq (c\beta)^\beta$, where c is an absolute constant. Therefore

$$m^\beta e_m(u) \leq (c\beta)^\beta \sup_{0<t\leq m} \left(t^\beta \min\big(1, \alpha A^{N/t}\big)\right).$$

Since the function $f(t) = t^\beta A^{N/t}$ is decreasing on the interval $(0, N(\ln A)/\beta]$ and $m \leq N(\ln A)/\beta$, the supremum above is attained for $t = N(\ln A)/\ln(1/\alpha)$. Thus

$$e_m(u) \leq \left(\frac{c\beta N \ln A}{m \ln(1/\alpha)}\right)^\beta \leq \left(\frac{2c\,(\ln(1/r))\,(\ln A)}{\eta \ln(1/\alpha)}\right)^{\ln(1/r)} \leq r$$

for $\alpha \leq A^{(2ec/\eta)\ln r}$. That proves the result. $\square$

Theorem 3.1 (with worse dependence of constants on δ) follows directly from Theorem 3.3 and Proposition 3.10 (see [LPRTV]), however application of Lemma 3.9 together with Proposition 3.10 and the standard volume covering estimates implies better dependence of constants on δ.

Proof of Theorem 3.1: The proof is similar to the proof of Theorem 3.3. First we split the Euclidean sphere into two parts and define corresponding events. Then for each event we will use covering of K by small balls reducing the problem of estimating of probability that a vector belongs to a multiple of K to the problem of estimating of probability that a vector belongs to a shift of the Euclidean ball, which we already considered. Note that in two different cases we will use different estimates for covering numbers, in one we apply Proposition 3.10 in the second the standard volume estimate $N(K, B_2^N) \leq (3V_K)^N$ is enough.

First we define the following two events corresponding to $\Omega(w, t, a, b)$ and $\Omega'(w, t, a, b)$. As above, fix some positive a, b and let $S(a, b)$, $S'(a, b)$ be the corresponding splitting of S^{n-1}. Set

$$\Omega_K(t, a, b) \;=\; \overline{\Omega} \cap \left(\exists\, x \in S(a, b) \text{ s.t. } \|\Gamma x\|_K \leq t\sqrt{N}\right), \qquad (3.14)$$

$$\Omega'_K(t, a, b) \;=\; \overline{\Omega} \cap \left(\exists\, x \in S'(a, b) \text{ s.t. } \|\Gamma x\|_K \leq t\sqrt{N}\right), \qquad (3.15)$$

where, as before, $\overline{\Omega} = \left\{\omega : \|\Gamma\| \leq a_1\sqrt{N}\right\}$.

Now let c_1, c_2, c_3, $\bar{c}$ be constants from Lemma 3.9. Let $V = 3eV_K$, $b = \min\{1/4,\ c_1/(a_1\mu^3)\}$, $t_0 = a_1 b/2$, $t = \min\left\{t_0\ ,\ \frac{b}{\bar{c}V}\left(\frac{b}{3\bar{c}a_1V}\right)^{1/\delta}\right\}$, $a = t/a_1$. Note that we can ensure that a satisfies the conditions of Lemma 3.9 by an appropriate choice of constant C_1 in the condition of Theorem 3.1.

Case I: Probability of $\Omega'_K(t, a, b)$. Since $N(K, B_2^N) \leq A := (3V_K)^N$ there are w_1, ..., w_A such that for every $z \in t\sqrt{N}K$ there exists $i \leq A$ such that $z \in w_i + t\sqrt{N}B_2^N$. Therefore, by Lemma 3.9,

$$\mathbb{P}\left(\Omega'_K(t, a, b)\right) \leq \sum_{i=1}^{A} \mathbb{P}\left(\Omega'(w_i, t, a, b)\right) \leq AV^{-N} = e^{-N}.$$

Case II: Probability of $\Omega_K(t,a,b)$. We apply Proposition 3.10 for

$$r = \min\left\{\frac{t_0}{a_1}, \frac{1}{e}\right\}, \ \eta = \min\left\{\frac{c_3}{2\mu^6}, \frac{\ln(4\pi V_K)}{\ln(1/r)}\right\}, \quad \text{and} \quad \alpha = (4\pi V_K)^{(c/\eta)\ln r}.$$

There exist w_1, ..., w_A, where $A \le \exp(c_3 N/(2\mu^6))$, such that for every $z \in a_1\sqrt{N}(\alpha K \cap B_2^N)$ there exists $i \le A$ such that $z \in w_i + ra_1\sqrt{N}B_2^N$. Since for every $\omega \in \overline{\Omega}$ and every $x \in S^{n-1}$ one has $\Gamma x \in a_1\sqrt{N}B_2^N$, applying Lemma 3.9, we obtain

$$\begin{aligned}
&\mathbb{P}\left(\Omega_K(\alpha a_1, a, b)\right) \\
&\le \sum_{i=1}^{A} \mathbb{P}\left(\Omega(w_i, ra_1, a, b)\right) \le \sum_{i=1}^{A} \mathbb{P}\left(\Omega(w_i, t_0, a, b)\right) \\
&\le A\exp(-c_3 N/\mu^6) \le \exp(-c_3 N/(2\mu^6)).
\end{aligned}$$

Finally, let $\gamma = \min\{t, \alpha a_1\}$. Then $\gamma > (C(a_1,\mu)/V_K)^{1+1/\delta}$ and

$$\begin{aligned}
\mathbb{P}\left(\Omega'_K(\gamma, a, b)\right) &\le \mathbb{P}\left(\Omega'_K(t, a, b)\right) \le e^{-N}, \\
\mathbb{P}\left(\Omega_K(\gamma, a, b)\right) &\le \mathbb{P}\left(\Omega_K(\alpha a_1, a, b)\right) \le \exp(-c_3 N/(2\mu^6)).
\end{aligned}$$

We conclude as in Theorem 3.3 obtaining that the desired probability does not exceed $e^{-N} + \exp(-c_3 N/(2\mu^6)) + \exp(-a_2 N)$. This completes the proof of Theorem 3.1. $\square$

We describe below an alternative approach to Proposition 3.10. Let $K \subset \mathbb{R}^N$ be a symmetric convex body. Recall the important definition of the M^*-functional,

$$M^*(K) := \int_{S^{N-1}} \sup_{y\in K} \langle x, y\rangle \ dx.$$

Now, consider the function $M_K^*(\cdot) : (0,\infty) \to [0,1]$ defined by $M_K^*(r) = M^*(L)$, where $L := (K/r) \cap B_2^N$. In [GM1], [GM2] many properties of K were investigated using the function $M_K^*(r)$. The following proposition provides estimates for this function in terms of V_K.

Proposition 3.11 *Let $K \subset \mathbb{R}^N$ be a symmetric convex body such that $B_2^N \subset K$. There exists an absolute constant C such that $M_K^*(r) \le C\sqrt{\frac{\ln(2V_K)}{\ln(1+r^2\ln(2V_K))}}$ for every $r > 0$. In particular, if $r \ge 2(2V_K)^{1/\eta}$ then $M_K^*(r) \le C\sqrt{\eta}$.*

Remark. By Sudakov's inequality this proposition implies Proposition 3.10 with $\alpha = (2V_K)^{-c/(\eta r^2)}$.

Proof: Note that for $r^2 \ln(2V_k) \le 1$ Proposition 3.11 is trivially satisfied, since $V_K \ge 1$ and $M_K^*(r) \le 1$. Hence we may assume that $r > 1/\sqrt{\ln(2V_k)}$.

Denote $M_K^*(r) = M^*(L)$ by M^*. Since $L \subset B_2^N$, by the dual version of Dvoretzky theorem, there exist an absolute constant $0 < c' < 1/4$ and a subspace $E \subset \mathbb{R}^N$ of dimension $k \ge c'(M^*)^2 N$ such that $P_E K \supset r P_E L \supset (rM^*/2) P_E B_2^N$. Here P_E denotes the orthogonal projection onto E. Since $K \supset B_2^N$, the Rogers-Shephard inequality (see [Pi] Lemma 8.8) implies

$$
\begin{aligned}
V_K &= \left(\frac{|K|}{|B_2^N|}\right)^{1/N} \ge \binom{N}{k}^{-1/N} \left(\frac{|P_E K|\,|K \cap E|}{|B_2^N|}\right)^{1/N} \\
&\ge \binom{N}{k}^{-1/N} \left(\frac{|(rM^*/2)B_2^k||B_2^{N-k}|}{|B_2^N|}\right)^{1/N} \ge \frac{1}{2}\left(\frac{rM^*}{2}\right)^{k/N}.
\end{aligned}
$$

Thus if $M^* > 2/r$ then $2V_K \ge (rM^*/2)^{c' M^{*2}}$, which implies

$$
M^{*2} \le 4 \ln(2V_K)/(c' \ln(r^2 \ln(2V_K))).
$$

Finally, if $M^* \le 2/r$, then the conclusion follows from the fact that $r > 1/\sqrt{\ln(2V_k)}$. □

As an application, we show how the last proposition implies a volume ratio result. Firstly, it was noticed in [GM2] that the well-known lower M^*-estimate ([Mi], [PT], [Go]) can be formulated as follows:

Proposition 3.12 *Let $\lambda, \varepsilon \in (0,1)$. Let n be large enough, $k = [\lambda n]$, and K be a symmetric convex body in $\mathbb{R}^n$. Assume r be such that $M_K^*(r) = (1-\varepsilon)\sqrt{1-\lambda}$. Then there exists a k-dimensional subspace E of $\mathbb{R}^n$ such that $K \cap E \subset rB_2^n$. Moreover, the measure (normalized Haar measure on the Grassman manifold $G_{n,k}$) of such subspaces is larger than $1 - \exp(-c_\varepsilon n)$, where c_ε depends only on ε.*

Now, combining Propositions 3.11 and 3.12 (for, say, $\varepsilon = 1 - 1/\sqrt{2}$) we immediately obtain

Theorem 3.13 *Let $\lambda \in (0,1)$. Let n be large enough, $k = [\lambda n]$, and K be a symmetric convex body in $\mathbb{R}^n$ such that $B_2^n \subset K$. Then there exists a k-dimensional subspace E of $\mathbb{R}^n$ such that*

$$K \cap E \subset 2(2V_K)^{C/(1-\lambda)}\ B_2^n,$$

where C is an absolute constant from Proposition 3.11. Moreover, the measure of such subspaces is larger than $1 - \exp(-cn)$, where c is a positive absolute constant.

Szarek's volume ratio theorem ([Sz], [SzT], see also [Pi]) stated under the assumptions of Theorem 3.13 is the same result with explicit constants. Namely, the set of all k-dimensional subspaces $E \subset \mathbb{R}^n$ satisfying

$$K \cap E \subset (4\pi V_K)^{1/(1-\lambda)}\ B_2^n,$$

has measure larger than or equal to $1 - 2^{-n}$.

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