Randomized Isomorphic Dvoretzky Theorem

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Abstract

Let K be a symmetric convex body in $\mathbb{R}^N$ for which B_2^N is the ellipsoid of minimal volume. We provide estimates for the geometric distance of a "typical" rank n projection of K to B_2^n, for $1 \leq n < N$. Known examples show that the resulting estimates are optimal (up to numerical constants) even for the Banach–Mazur distance.

Résumé

French title here

Soit K un corps convexe symétrique de $\mathbb{R}^N$ dont l'ellipsoïde de volume minimal le contenant est la boule euclidienne B_2^N. Nous estimons la distance géométrique de projections "typiques" de rang n de K à la boule B_2^n pour tout $n \in \{1, \ldots, N-1\}$ (i.e. nous prouvons qu'il en existe une vaste proportion au sens de la mesure de Haar sur la grassmanienne). Des exemples bien connus permettent de dire que ces estimations sont optimales (à des constantes numériques près), même pour la distance de Banach–Mazur.

Précédemment, des résultats sur l'existence de projections orthogonales de K (ou d'un vaste ensemble de telles projections si l'on ajoute une restriction sur leur rang) avec un bon contrôle de la distance à la boule euclidienne ont été prouvés dans [11] et [13], et dans un cadre gaussien dans [4] et [3]. Dans cette note, nous donnons une preuve unifiée de l'existence de projections orthogonales "typiques" pour toutes les valeurs de la dimension n.

Comparés aux précédents arguments connus, notre preuve repose sur deux nouveaux ingrédients. Tout d'abord, nous utilisons un résultat de [7] qui permet de réduire l'étude du rayon d'une boule euclidienne inscrite dans des projections aléatoires d'un corps convexe à la même question posée pour une boule euclidienne inscrite dans une section fixée du corps convexe. Le second ingrédient est un résultat de Vershynin [15] qui nous permet de trouver un sous-ensemble de points de contact équivalent à un système orthonormal.

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The main result of this note is Theorem 1 below, which generalizes various extensions of Dvoretzky's theorem proved by Milman and Schechtman in [11], [13] (see also [3], [4], [6]). It provides estimates for radii of Euclidean balls inscribed and superscribed into a "typical" projection of a symmetric convex body K with respect to the natural Euclidean structure induced by the ellipsoid of minimal volume. The existence (rather than "typicality") of projections with good control of the radii was proved in [13], and the result for "typical" projections was proved in [11] and [13] for a limited range of dimensions. In this note we provide a uniform proof for typical projections over the full range of dimensions.

Similarly as in [11] our proof follows a general line of [8], and compared with previous arguments it is based on two new ingredients. Firstly, we use a result from [7] which reduces the estimates for radii of a Euclidean ball inscribed in random projections of a convex body to a similar question inside any fixed section of the body ((∗) below). The second fact is a result of Vershynin [15] which provides a subset of contact points well equivalent to an orthonormal system ((∗∗) below).

We also prove, in Proposition 4, a related simple optimization fact which states that the regular cube has the maximal Gaussian measure among all parallelopipeds inscribed in a Euclidean ball. This can be viewed as a complementary result to Gluskin's minimization theorem [1], which was also closely related to the Gluskin's lemma used in [13].

The example of the unit ball in ℓ_1^n shows that the estimates in Theorem 1 are optimal (up to numerical constants), even if the "geometric distance" is replaced by the smaller Banach–Mazur distance to the Euclidean ball.

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For $N \geq 1$, by $\{e_i\}_{i=1}^N$ we denote the canonical vector basis in $\mathbb{R}^N$, equipped with the Euclidean norm $\|\cdot\|_2$. The unit ball with respect to $\|\cdot\|_2$ will be denoted by B_2^N, while B_∞^N will stand for the set of all $x = \sum t_i e_i \in \mathbb{R}^N$ such that $|t_i| \leq 1$ for all $1 \leq i \leq N$. $G_{N,k}$ will be the set of all k dimensional subspaces of $\mathbb{R}^N$ equipped with the normalized Haar measure μ. For $H \in G_{N,k}$, by P_H we denote the orthogonal projection onto H. For a symmetric convex body K in $\mathbb{R}^N$ we let $k^*(K)$ to be the maximal dimension k such that for some constant $a > 0$ we have

$$\mu\big\{H \in G_{N,k} \mid aP_H(B_2^N) \subset P_H(K) \subset 4aP_H(B_2^N)\big\} \geq 1 - e^{-k},$$

cf., [12], [9], and [7] (1.2) for the measure concentration used here.

Theorem 1 *There exist constants $c, c' > 0$ such that for every $N \geq 1$ and every symmetric convex body K in $\mathbb{R}^N$ for which the unit Euclidean ball B_2^N is the ellipsoid of minimal volume containing K, and every $k^*(K) \leq n \leq N$, we have*

$$\mu\{H \in G_{N,n} \mid c\sqrt{\log(N/n)/N} P_H B_2^N \subset P_H(K) \subset c^{-1}\sqrt{n/N} P_H B_2^N\} \geq 1 - e^{-c'n}.$$

The existence of an $H \in G_{N,n}$ satisfying the inclusions defining the set above was proved in [13], and Theorem 1 in the range $1 \leq n \leq cN/\log N$ was proved in [11] and [13] (see the Remark on page 162 of [13]). Here we provide a uniform proof of the measure estimates over the full range of dimensions.

Clearly, by the definition of k^*, $P_H(K)$ concentrates around $aP_H(B_2^N)$, for $1 \leq k \leq k^*(K)$. For $k \geq k^*(K)$, the fact that the right hand side inclusion holds with probability $\geq 1 - e^{-c'n}$ is the well known "shrinking" principle for the diameter of random projections of an arbitrary convex

body, *cf.,* [9] (also see [7], Proposition 4.3 for the formulation). The proof of the fact that the left hand side inclusion in the statement of Theorem 1 holds with probability $\geq 1 - e^{-c'n}$ depends in an essential way on a result from [7], Proposition 3.2 which yields:

$(*)$*There exists a constant* $\bar{c} > 0$ *such that for an arbitrary symmetric convex body* K_0 *in an* $N/2$ *dimensional subspace* F *of* R^N*, and for every* $n \leq N/4$ *we have*

$$\mu\{H \in G_{N,n} \mid P_H(K_0) \supset \bar{c} P_H B_2^N\} \geq \mu\{H \in G_{F,n} \mid P_H(K_0) \supset P_H B_2^N\} - e^{-\bar{c}n}.$$

Therefore, for $n \leq \tilde{c}N$ setting $K_0 = K \cap F$ the left hand side inclusion is a formal consequence (with different constants) of the following statement. The case $n > \tilde{c}N$ is trivial.

Proposition 2 *There exist constants* $c, c' \geq 0$ *and* $0 < \tilde{c} < 1/4$ *such that for every* $K \subset \mathbb{R}^N$ *as in Theorem 1 there exists a* $N/2$ *dimensional subspace* $F \subset \mathbb{R}^N$ *such that for every* $1 \leq k' \leq \tilde{c}N$ *we have*

$$\mu\{H \in G_{F,k'} \mid P_H(K \cap F) \supset c\sqrt{\log(N/k')/N} P_H B_2^N\} \geq 1 - e^{-c'k'}.$$

The choice of a subspace F will depend on the existence of contact points of the body K and B_2^N which are well equivalent to an orthonormal system. Recall that vectors $\{z_i\}_{i=1}^m$ in $\mathbb{R}^m$ are said to satisfy a lower ℓ_2 estimate with constant $A_1 > 0$ and an upper ℓ_2 estimate with constant A_2 provided that

$$(1/A_1)\Big(\sum_{i=1}^m t_i^2\Big)^{1/2} \leq \Big\|\sum_{i=1}^m t_i z_i\Big\|_2 \leq A_2\Big(\sum_{i=1}^m t_i^2\Big)^{1/2},$$

for all scalars $t_1, \ldots, t_m$.

Another crucial ingredient in our proof is a consequence of a recent result of Vershynin [15], Corollary 5.1, which can be formulated as follows:

$(**)$ *For every* K *as in Theorem 1 there are* $m \geq N/2$ *contact points* $z_1, \ldots, z_m$ (i.e., $\|z_i\|_K = \|z_i\|_{K^0} = \|z_i\|_2 = 1$ *for* $i = 1, \ldots, m$*) which satify lower and upper* ℓ_2 *estimates with constants* $A_1 = A_2 = C$*, where* $C > 0$ *is a numerical constant.*

Thus Theorem 1 reduces to

Proposition 3 *Let* $m \geq 1$ *and let* $\{z_i\}_{i=1}^m$ *in* $\mathbb{R}^m$ *satisfy lower and upper* ℓ_2 *estimates with constants* A_1 *and* A_2*, respectively. There exist universal constants* $c, c' > 0$ *and* $\tilde{c} = \tilde{c}(A_1, A_2) > 0$ *depending on* A_1, A_2 *only such that for every* $1 \leq k' \leq \tilde{c}m$ *we have*

$$\mu\Big\{H \in G_{m,k'} \mid \sup_{1 \leq i \leq m} |(x, z_i)| \geq c\sqrt{\log(m/k')/m} \quad \text{for } x \in S^{m-1} \cap H\Big\} \geq 1 - e^{-c'k'}. \tag{1}$$

Indeed, assuming the validity of Proposition 3, Proposition 2 can be proved for $F = \operatorname{span}[z_i]_{i=1}^m$. Identify this F with $\mathbb{R}^m$. Since $K \cap F \supset \operatorname{conv}(\pm z_i)_{i=1}^m$, it is easy to see by duality that for a given $H \in G_{m,k'}$ the condition $\sup_{1 \leq i \leq m} |(x, z_i)| \geq c\sqrt{\log(m/k')/m}$ for all $x \in S^{m-1} \cap H$ implies the inclusion $P_H(K \cap F) \supset c\sqrt{\log(N/k')/N} P_H B_2^N$. Thus Proposition 2 follows.

Proof of Proposition 3 Fix $1 \leq k \leq m$ and a partition $\{\sigma_j\}_{j=1}^k$ of $\{1, 2, \ldots, m\}$ into mutually disjoint subsets with $[m/k] \leq |\sigma_j| \leq [m/k] + 1$ for all $1 \leq j \leq k$. Define the norm $|||\cdot|||$ on $\mathbb{R}^m$ by

$$|||x||| = (1/k) \sum_{j=1}^k \sup_{i \in \sigma_j} |(x, z_i)| \quad \text{for} \quad x \in \mathbb{R}^m. \tag{2}$$

We prove an estimate analogous to (1) for the norm $|||\cdot|||$, and we shall work in the Gaussian setting. Recall that a random vector $g \in I\!R^m$ of the form $g = \sum_{i=1}^m \tilde{\gamma}_i e_i$, where $\tilde{\gamma}_i$ are independent $N(0,1)$ distributed Gaussian variables, is called a standard Gaussian vector. The usual approach requires a lower estimate for the expected value $I\!E|||g|||$, and an upper estimate for the Lipschitz constant of $|||\cdot|||$.

Let $\{z_i'\}_{i=1}^m$ in $I\!R^m$ be the biorthogonal system to $\{z_i\}_{i=1}^m$; thus $\{z_i'\}_{i=1}^m$ satisfies lower and upper ℓ_2 estimates with constants A_2 and A_1, respectively. It easily follows from the lower ℓ_2 estimate for $\{z_i'\}$'s that $|||\cdot|||$ satisfies the Lipschitz condition with respect to the Euclidean norm with constant $A_2/\sqrt{k}$. First we show that there is an absolute constant $c_1 > 0$ such that

$$I\!E|||g||| \geq (c_1/A_1)\sqrt{\log(1+m/k)}. \tag{3}$$

Observe first that for any $1 \leq \ell \leq m$ we have

$$I\!E \sup_{1\leq i\leq \ell} |(g, z_i)| \geq (c_1/A_1)\sqrt{\log(\ell+1)}, \tag{4}$$

where $c_1 > 0$ is an absolute constant. Indeed, by Sudakov's minoration theorem (*cf., e.g.,* [14], Theorems 5.5 and 5.6) we have

$$I\!E \sup_{1\leq i\leq \ell} |(g, z_i)| \geq c_1 \sup_{\varepsilon>0} \varepsilon\sqrt{\log(1+N(\varepsilon))},$$

where for a given $\varepsilon > 0$, by $N(\varepsilon)$ we denote the covering number $N(\mathrm{conv}\,\{\pm z_i\}_{1\leq i\leq \ell}, \varepsilon B_2^m)$, *i.e.,* the smallest number N of balls εB_2^m which shifts can cover the set $\mathrm{conv}\,\{\pm z_i\}_{1\leq i\leq \ell}$. Since $\|z_i - z_{i'}\| \geq \sqrt{2}/A_1$ for all $1 \leq i, i' \leq \ell$, $i \neq i'$, (4) immediately follows (with a different constant c_1). Then (3) follows by the definition (2) of $|||\cdot|||$.

We give a few more details of Proposition 3. Let $k' = [\alpha(c_1/A_1A_2)^2 k]$, where $0 < \alpha < 1$ will be determined later. Let Γ denote an $m \times k'$ matrix with independent Gaussian $N(0,1)$ entries. We shall treat Γ as the operator $\Gamma : I\!R^{k'} \to I\!R^m$.

Fix $\beta > 0$ sufficiently small. Since the Lipschitz constant of $|||\cdot|||$ is less than or equal to $A_2/\sqrt{k}$, by the standard Gaussian deviation inequality we get, for all $x \in S^{k'-1}$

$$I\!P\Big\{ \big| |||\Gamma x||| - I\!E|||\Gamma x||| \big| \leq (\beta c_1/A_1)\sqrt{\log(1+k/m)} \Big\} \geq 1 - 2(k/m)^{(\beta c_1/A_1A_2)^2 k/2}.$$

Combining (3) with a standard ε-net argument (*cf., e.g.,* [10]), and chosing a suitable $\varepsilon > 0$, we obtain

$$I\!P\Big\{ |||\Gamma x||| \geq (\beta c_1/2A_1)\sqrt{\log\,(1+m/k)} \text{ for } x \in S^{k'-1} \Big\} \geq 1 - e^{-k'}, \tag{5}$$

provided that $k \leq \delta m < m$ and we chose $\alpha > 0$ to satisfy $(3/\varepsilon)^{-k'} 2\delta^{(\alpha c_1/A_1A_2)^2 k/2} \leq e^{-k'}$.

Since $\sup_{1\leq i\leq m} |(x, z_i)| \geq |||x|||$ for all $x \in I\!R^m$, (1) follows from (5) by a standard argument. □

Remark The proof above does not use the full strength of the lower ℓ_2 estimate, but only a weaker fact that the vectors $\{z_i\}$ are well separated.

We next prove a related result which can also provide an alternative proof of (3). Namely, using Proposition 4 one easily shows that if $\{z_i\}$ are as in Proposition 3 then we have $I\!P\{\sup_{1\leq i\leq \ell} |(g, e_i)| > t\} \leq I\!P\{\sup_{1\leq i\leq \ell} |(g, z_i)| > t/A_1\}$, for all $t > 0$. Proposition 4 is a natural maximization result which might be of an independent interest, and we provide a sketch of its proof.

Proposition 4 *Let $n \geq 1$ and let g be a standard Gaussian vector in $\mathbb{R}^n$. Let $y_1, \ldots, y_n \in \mathbb{R}^n$ and let $Q = \sum_{i=1}^n [-y_i, y_i]$ be the parallelopiped generated by the y_i's. If $Q \subset \sqrt{n} B_2^n$ then for an arbitrary $t > 0$ we have $\mathbb{P}\{g \in tQ\} \leq \mathbb{P}\{g \in tB_\infty^n\}$.*

Proof It suffices to prove that for every $a > 0$, the inclusion $Q \subset aB_2^n$ implies $\mathbb{P}\{g \in Q\} \leq \mathbb{P}\{g \in (a/\sqrt{n})B_\infty^n\}$. To this end set $Q^* = \sum_{i=1}^{n-1} [-y_i, y_i]$. Clearly we may assume that $Q^* \subset \mathbb{R}^{n-1} = \mathrm{span}\,[e_i \mid 1 \leq i \leq n-1]$.

Write y_n in the form $y_n = v + te_n$, where $v \in \mathbb{R}^{n-1}$. Replacing y_n by $-y_n$ if needed we may assume that $t > 0$. Let y be any vertex of Q^*. By the parallelogram identity applied to y and y_n, using the fact that $Q \subset aB_2^n$, we get $a^2 \geq (\|y + y_n\|_2^2 + \|y - y_n\|_2^2)/2 = \|y\|_2^2 + \|v\|_2^2 + t^2$. Hence $Q^* \subset \sqrt{a^2 - t^2} B_2^{n-1}$.

For $k \geq 1$ by γ_k denote the standard Gaussian measure on $\mathbb{R}^k$, so that we have $\gamma_k(B) = \mathbb{P}\{g \in B\}$ for every Borel subset B of $\mathbb{R}^k$. For $s \in \mathbb{R}$, let $\mathbb{R}_s^{n-1} = \mathbb{R}^{n-1} + se_n$, and let $\gamma_{n-1,s}(B + se_n) = \gamma_{n-1}(B)$ for every Borel subset B of $\mathbb{R}^{n-1}$. Observe that for $|s| \leq t$ we have $Q \cap \mathbb{R}_s^{n-1} = Q^* + (s/t)y_n = (Q^* + se_n) + (s/t)v$. Thus, by Anderson's inequality (*cf.*, *e.g.*, [5], Sec. 11, Th. 9), we have

$$\gamma_{n-1,s}(Q \cap \mathbb{R}_s^{n-1}) \leq \gamma_{n-1,s}(Q^* + se_n) = \gamma_{n-1}(Q^*). \tag{6}$$

The proof of the proposition will be completed by induction. For $n = 1$ the statement is trivial. Assume the statement for $n - 1$. Since $Q^* \subset \sqrt{a^2 - t^2} B_2^{n-1}$, by the induction hypothesis and (6), for every $|s| \leq t$ we get $\gamma_{n-1,s}(Q \cap \mathbb{R}_s^{n-1}) \leq \gamma_{n-1}(bB_\infty^{n-1})$, where $b = \sqrt{(a^2 - t^2)/(n-1)}$. Thus

$$\gamma_n(Q) = \int_{-t}^{t} \gamma_{n-1,s}(Q \cap \mathbb{R}_s^{n-1})\, e^{-s^2/2}\, \frac{ds}{\sqrt{2\pi}} \leq (2\pi)^{-n/2} \int_{-t}^{t} e^{-s^2/2}\, ds \left(\int_{-b}^{b} e^{-s^2/2}\, ds \right)^{n-1}. \tag{7}$$

Consider the function $f(h) = \int_0^h e^{-s^2/2} ds$ for $h \geq 0$ and note that f is increasing while f' is decreasing. The inequality (7) can be expressed as

$$\gamma_n(Q) \leq (2/\pi)^{n/2} f(a\cos\alpha) f^{n-1}((a\sin\alpha)/\sqrt{n-1}) =: \Phi(\alpha), \tag{8}$$

where $\alpha \in [0, \pi/2]$ is defined by the equality $\cos\alpha = t/a$ (note that clearly $t \leq a$). Now, using the fact that the functions f' and $\cos\alpha$ are decreasing while the functions f and $\sin\alpha$ are increasing we deduce that Φ' has only one zero in $(0, \pi/2)$, which is attained at the point β satisfying $\sin\beta/\sqrt{n-1} = \cos\beta$. This yields $\sin\beta = \sqrt{(n-1)/n}$ and $\cos\beta = 1/\sqrt{n}$. Since Φ is nonnegative and $\Phi(0) = \Phi(\pi/2) = 0$ we infer that Φ attains its maximum at β. In view of (8) the proof is completed by observing that $\Phi(\beta) = \gamma_n((a/\sqrt{n})B_\infty^n)$. □

Remark The results of [11] and [13] were extended to the non-symmetric case by two different approaches in [3] and [6] (with the former paper giving randomness of projections for the limited range of ranks). In the present formulation Theorem 1 holds for nonsymmetric body K and $n \leq \alpha N$ for some $0 < \alpha < 1$ (and hence for $n \leq \beta N$ for every $0 < \beta < 1$, with constants depending on β). Indeed, one can check that $(*)$ and $(**)$ remain valid for nonsymmetric bodies. Our argument can be adopted to the nonsymmetric case by replacing $|(x, z_i)|$ with $(x, z_i) \vee 0$ in the definition of $|||\cdot|||$.

Remark In [4] and [3] techniques developed in [2] were used to obtain an upper bound for the minimal Banach-Mazur distance of an n-dimensional section of an arbitrary N-dimensional convex body to the n-dimensional Euclidean ball, similar as in [11] and [13]. The same techniques may be used in our context as well.

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