

Asymmetry of convex polytopes and vertex index of symmetric convex bodies. *

E. D. Gluskin A. E. Litvak

Abstract

In [GL] it was shown that a polytope with few vertices is far from being symmetric in the Banach-Mazur distance. More precisely, it was shown that Banach-Mazur distance between such a polytope and any symmetric convex body is large. In this note we introduce a new, averaging-type parameter to measure the asymmetry of polytopes. It turns out that, surprisingly, this new parameter is still very large, in fact it satisfies the same lower bound as the Banach-Mazur distance. In a sense it shows the following phenomenon: *if a convex polytope with small number of vertices is as close to a symmetric body as it can be, then most of its vertices are as bad as the worst one.* We apply our results to provide a lower estimate on the vertex index of a symmetric convex body, which was recently introduced in [BL]. Furthermore, we give the affirmative answer to a conjecture by K. Bezdek [B3] on the quantitative illumination problem.

1 Introduction

Let \mathbf{K} be a convex body in \mathbb{R}^d . How asymmetric is \mathbf{K} ? More precisely, how far is \mathbf{K} from being centrally symmetric? Of course, such a question requires a functional, which measures asymmetry, or a distance between convex bodies. One of the natural ways to introduce such a measure is the following functional

$$d_{\mathbf{K}} := \inf_{a \in \mathbf{K}} \sup_{x \in \mathbf{K}-a} \| -x \|_{\mathbf{K}-a},$$

*Keywords: asymmetry of polytopes, illumination parameter, measures of symmetry, polytopes with few vertices, vertex index. 2000 Mathematical Subject Classification. Primary: 52B11, 52A20, 52B15. Secondary: 51M16, 52A40, 46B07

where $\|\cdot\|_{\mathbf{K}}$ denotes *the Minkowski functional* of \mathbf{K} or *the gauge* of \mathbf{K} (see the definitions below). This functional is closely related (in fact, is equivalent to) the Banach-Mazur distance between \mathbf{K} and the set of all centrally symmetric bodies in \mathbb{R}^d . We refer to [Gr] for related discussion on this and other ways to measure the asymmetry of a given convex body in \mathbb{R}^d .

In the present note we deal with convex d -polytopes, i.e. with convex polytopes whose interiors are not empty. Let \mathbf{K} be such a polytope and $\{x_i\}_{i=1}^m$ be its vertices. It is easy to see that $d_{\mathbf{K}} = \max_i \|\cdot - (x_i - a)\|_{\mathbf{K}-a}$ for some $a \in \mathbf{K}$. Thus, the functional $d_{\mathbf{K}}$ takes into account only one, the worst, vertex of \mathbf{K} . We suggest another, averaging-type functional to measure the asymmetry of convex polytopes, namely we define

$$\delta_{\mathbf{K}} := \inf_{a \in \mathbf{K}} \frac{1}{m} \sum_{i=1}^m \|\cdot - (x_i - a)\|_{\mathbf{K}-a}.$$

Thus, our functional measures how bad vertices are in average (“ x is bad” here means that $\|\cdot - x\|_{\mathbf{K}}$ is big, which in turn means that $-x$ is far away from the body). Clearly, $\delta_{\mathbf{K}} \leq d_{\mathbf{K}}$. Our main result, Theorem 3.2, states:

Let $1 \leq k \leq d$ and $m = d + k$. Let \mathbf{K} be a convex d -polytope in \mathbb{R}^d with m vertices x_1, x_2, \dots, x_m . Then

$$\delta_{\mathbf{K}} \geq \frac{m}{2k}.$$

It shows that a convex d -polytope cannot be centrally symmetric unless it has at least $2d$ vertices and provides a quantitative lower bound. The bound should be compared with the main result from [GL], which states that $d_{\mathbf{K}} \geq d/k$ and which is sharp (i.e., for every d, k there exists a convex d -polytope in \mathbb{R}^d with $d + k$ vertices such that $d_{\mathbf{K}} \leq \lceil d/k \rceil$, where $\lceil a \rceil$ denotes the smallest integer larger than or equal to a). Thus, in general, the lower bound for the *worst vertex* is almost the same as the lower bound for the *average vertex*. In other words, Theorem 3.2 discovers the following (high-dimensional) phenomenon: *if a convex polytope with small number of vertices is as close to a symmetric body as it can be, then most of its vertices are as bad as the worst one.*

We apply our results to provide sharp lower bounds for the vertex index of a centrally symmetric convex body \mathbf{K} , $\text{vein}(\mathbf{K})$, and for the illumination parameter of a centrally symmetric convex body \mathbf{K} , $\text{ill}(\mathbf{K})$ (see the precise definitions in Section 4). Both quantities are closely connected to some

important quantities and problems in asymptotic theory of normed spaces and in convex geometry, including the problem of covering a convex body by smaller homothetic copies of itself. In particular, the illumination parameter of a convex body was introduced in [B1] as a cost function for the Boltyanski-Hadwiger illumination conjecture. We refer the interested reader to [B2, B3, BL, MS] and references therein for more details.

K. Bezdek conjectured (see Problem 6.3 of [B3]) that for every centrally symmetric d -dimensional body \mathbf{K} one has

$$\text{ill}(\mathbf{K}) \geq 2d.$$

Applying our results on the asymmetry of polytopes, in Theorem 4.1 we prove that for every d -dimensional centrally symmetric convex body \mathbf{K} one has

$$\text{vein}(\mathbf{K}) \geq 2d.$$

As $\text{ill}(\mathbf{K}) \geq \text{vein}(\mathbf{K})$, it proves Bezdek's conjecture.

2 Preliminaries and Notation

By $|\cdot|$ and $\langle \cdot, \cdot \rangle$ we denote the canonical Euclidean norm and the canonical inner product on \mathbb{R}^d . The canonical basis of \mathbb{R}^d we denote by e_1, \dots, e_d . Given points x_1, \dots, x_k in \mathbb{R}^d we denote their convex hull by $\text{conv} \{x_i\}_{i \leq k}$.

Let $\mathbf{K} \subset \mathbb{R}^d$ be a compact convex body with nonempty interior such that $0 \in \mathbf{K}$. We denote by \mathbf{K}° the polar of \mathbf{K} , i.e.,

$$\mathbf{K}^\circ = \{x \mid \langle x, y \rangle \leq 1 \text{ for every } y \in \mathbf{K}\}.$$

In particular, if 0 is on the boundary of \mathbf{K} then \mathbf{K}° is unbounded star-shaped convex set. We will also use that if E is a linear subspace of \mathbb{R}^d then the polar of $\mathbf{K} \cap E$ (taken in E) is

$$(\mathbf{K} \cap E)^\circ = P_E \mathbf{K}^\circ,$$

where P_E is the orthogonal projection onto E .

The Minkowski functional (or the gauge) of a convex body \mathbf{K} containing 0 in its interior is defined as

$$\|x\|_{\mathbf{K}} = \inf\{\lambda > 0 \mid x \in \lambda \mathbf{K}\}.$$

Note that if \mathbf{K} is centrally symmetric with respect to the origin then $\|\cdot\|_{\mathbf{K}}$ is a norm on \mathbb{R}^d and \mathbf{K} is its unit ball. Below it will be convenient to consider the Minkowski functional even for unbounded closed star-shaped sets as well as for sets containing 0 as a boundary point under usual agreement $\inf \emptyset = \infty$. In particular, it can happen that $\|x\|_{\mathbf{K}} = 0$ for $x \neq 0$ or that $\|x\|_{\mathbf{K}} = \infty$.

Given real number a by $[a]$ we denote the largest integer not exceeding a and by $\lceil a \rceil$ we denote the smallest integer which is not smaller than a .

We also will use the following proposition, which is a combination of Weil's Theorem ([W], see also Theorem 2.3.1 of [GoK]) and Stinespring's result ([S], see also 3.7.8 in [GoK]). For the reader's convenience we outline the proof.

Proposition 2.1 *Let $T = \{t_{ij}\}$ be an $m \times m$ matrix and $\lambda_1, \lambda_2, \dots, \lambda_m$ be eigenvalues of T . Then*

$$\sum_{j=1}^m |\lambda_j| \leq \sum_{i,j=1}^m |t_{ij}|.$$

Proof: Recall that singular numbers of T are defined as

$$s_j(T) = \lambda_j \left((T^*T)^{1/2} \right),$$

where $\lambda_j(A)$ denotes j th largest eigenvalue of A (counted according to multiplicities). Weil's Theorem says that

$$\sum_{j=1}^m |\lambda_j(T)| \leq \sum_{j=1}^m s_j(T).$$

It is well known that $\gamma(T) := \sum_{j=1}^m s_j(T)$ is a norm (see, e.g., Theorem 3.7.1 in [GoK]). Therefore,

$$\gamma(T) \leq \sum_{i,j=1}^m |t_{ij}| \gamma(E_{ij}),$$

where E_{ij} denotes the matrix with zero entries except one element in i th row, j th column, which is 1. It is easy to see that $\gamma(E_{ij}) = 1$ for every i and j , which completes the proof. \square

3 Asymmetry of polytopes

Lemma 3.1 *Let $\Lambda = \{\lambda_{ij}\}$ be an $m \times m$ matrix of rank k with nonnegative entries such that $\lambda_{ii} \geq 1$ for every $i \leq m$. Then*

$$\sum_{i,j=1}^m \lambda_{ij} \geq 3m - 2k.$$

Moreover, if $m \geq 2k$ then

$$\sum_{i,j=1}^m \lambda_{ij} \geq m + \frac{m(m-1)}{2k-1}.$$

Remark. Note that the estimates of Lemma 3.1 are asymptotically sharp. Indeed, consider a block-diagonal matrix with k blocks $[m/k] \times [m/k]$ or $[m/k] \times [m/k]$ of rank one, such that each block has entries 1 only. Then we have

$$\sum_{i,j=1}^m \lambda_{ij} \leq \frac{m^2}{k} + \frac{k}{4}.$$

Proof of Lemma 3.1. First let us note that without loss of generality we can assume that $\lambda_{ii} = 1$ for every i (otherwise we pass to the matrix $\{\lambda_{ij}/\lambda_{ii}\}_{i,j}$).

Consider $T = \Lambda - I$, where I is the identity and denote its entries by t_{ij} . Clearly, $t_{ij} \geq 0$ and $t_{ii} = 0$ for every i, j . By λ_j denote the eigenvalues of T .

Since Λ is of rank k , at least $m - k$ of eigenvalues of T are equal to -1 (indeed, $T = -I$ on $\text{Ker}\Lambda$). Since

$$0 = \sum_{i=1}^m t_{ii} = \text{Trace}T = \sum_{i=1}^m \lambda_i,$$

we obtain

$$\sum_{i=1}^m |\lambda_i| \geq 2m - 2k.$$

Proposition 2.1 implies

$$\sum_{i,j=1}^m t_{ij} \geq 2m - 2k,$$

which shows

$$\sum_{i,j=1}^m \lambda_{ij} \geq 3m - 2k.$$

For $m < 2k$ it proves the result.

Now we assume that $m \geq 2k$. Let $\sigma \subset \{1, 2, \dots, m\}$ be of cardinality l for some $2k \leq l \leq m$. Let

$$\bar{\Lambda} = \{\lambda_{ij}\}_{i,j \in \sigma}.$$

Clearly the rank of $\bar{\Lambda}$ does not exceed k , so, by the first part, we have

$$\sum_{i,j \in \sigma} \lambda_{ij} \geq 3l - 2k.$$

Using averaging argument, we obtain

$$\begin{aligned} \sum_{i,j=1}^m \lambda_{ij} &= m + \sum_{\substack{i,j=1 \\ i \neq j}}^m \lambda_{ij} = m + \binom{m-2}{l-2}^{-1} \sum_{\substack{\sigma \subset \{1,2,\dots,m\} \\ |\sigma|=l}} \sum_{\substack{i,j \in \sigma \\ i \neq j}} \lambda_{ij} \\ &\geq m + \binom{m-2}{l-2}^{-1} \binom{m}{l} (2l - 2k) = m + 2 \frac{m(m-1)}{l(l-1)} (l-k). \end{aligned}$$

The choice $l = 2k$ completes the proof. \square

Theorem 3.2 *Let $1 \leq k \leq d$ and $m = d + k$. Let \mathbf{K} be a convex d -polytope in \mathbb{R}^d with m vertices x_1, x_2, \dots, x_m . Then*

$$\sum_{i=1}^m \| -x_i \|_{\mathbf{K}} \geq \frac{m^2}{2k} \geq \max \left\{ 2d, \frac{md}{2k} \right\}.$$

Remark 1. Since a shift of a convex d -polytope with m vertices is still a convex d -polytope with m vertices, Theorem 3.2 implies immediately

$$\delta_{\mathbf{K}} \geq \frac{m}{2k}.$$

Remark 2. As was noticed in [GL], the estimate is asymptotically sharp. Geometrically one can consider the following example. Let

$$\mathbb{R}^d = \bigoplus_{i=1}^k E_i,$$

where E_i 's are orthogonal coordinate subspaces of \mathbb{R}^d of dimension $\lceil d/k \rceil$ or $\lfloor d/k \rfloor$. In each E_i consider the regular simplex, denoting its vertices by x_{ij} , $j \leq \dim E_i + 1$ and set $\mathbf{K} = \text{conv} \{x_{ij}\}_{ij}$. Clearly,

$$\| -x_{ij} \| = \dim E_i.$$

Therefore

$$\frac{1}{m} \sum_{i,j} \| -x_{ij} \| \leq \max_{i,j} \| -x_{ij} \| = \max_i \dim E_i = \lceil d/k \rceil,$$

which means $\delta_{\mathbf{K}} \leq d_{\mathbf{K}} \leq \lceil d/k \rceil$.

Proof of Theorem 3.2. Consider the linear operator $T : \mathbb{R}^m \rightarrow \mathbb{R}^d$ defined by $Te_i = x_i$. Denote the kernel of T by L . Clearly, L is a k -dimensional subspace of \mathbb{R}^m . The orthogonal projection onto L^\perp we denote by P .

Since $\mathbf{K}^\circ = \{f \in \mathbb{R}^d \mid \langle f, x_j \rangle \leq 1 \text{ for every } j \leq m\}$, we have

$$A := \sum_{i=1}^m \| -x_i \|_{\mathbf{K}} = \sum_{i=1}^m \sup \{ \langle f, -x_i \rangle \mid f \in \mathbb{R}^d, \langle f, x_j \rangle \leq 1 \text{ for every } j \leq m \}.$$

Using $\langle f, x_i \rangle = \langle f, Te_i \rangle = \langle T^* f, e_i \rangle$, we obtain

$$A = \sum_{i=1}^m \sup \{ \langle h, -e_i \rangle \mid h \in \mathbb{R}^m, h \in L^\perp, \langle h, e_j \rangle \leq 1 \text{ for every } j \leq m \}.$$

Now denote

$$S := \{h \in \mathbb{R}^m \mid \langle h, e_j \rangle \leq 1 \text{ for every } j \leq m\}$$

and for every $i \leq m$ denote

$$Q_i := \{h \in \mathbb{R}^m \mid \langle h, e_i \rangle \geq -1\}.$$

Then

$$S^\circ = \{h \in \mathbb{R}^m \mid 0 \leq \langle h, e_j \rangle \text{ for every } j \leq m, \text{ and } \sum_{j=1}^m \langle h, e_j \rangle \leq 1\}$$

and

$$Q_i^\circ = \{h \in \mathbb{R}^m \mid -1 \leq \langle h, e_i \rangle \leq 0, \langle h, e_j \rangle = 0 \text{ for } j \neq i\}.$$

Therefore

$$\|z\|_{S^\circ} := \begin{cases} \sum_{j=1}^m \langle z, e_j \rangle & \text{if } \langle z, e_j \rangle \geq 0 \text{ for every } j \leq m, \\ \infty & \text{otherwise,} \end{cases}$$

which implies

$$\begin{aligned} \|z\|_{PS^\circ} &= \inf_{y \in L} \|z + y\|_{S^\circ} \\ &= \inf \left\{ \sum_{j=1}^m \langle z + y, e_j \rangle \mid y \in L, \langle y, e_j \rangle \geq -\langle z, e_j \rangle \text{ for every } j \leq m \right\}. \end{aligned}$$

Using duality and our notation, we observe

$$\begin{aligned} A &= \sum_{i=1}^m \sup_{h \in S \cap L^\perp} \langle h, -e_i \rangle = \sum_{i=1}^m \sup_{h \in S \cap L^\perp} \|h\|_{Q_i} \\ &= \sum_{i=1}^m \sup_{h \in Q_i^\circ} \|h\|_{PS^\circ} = \sum_{i=1}^m \| -e_i \|_{PS^\circ} \\ &= \sum_{i=1}^m \inf \left\{ \sum_{j=1}^m \langle y, e_j \rangle - 1 \mid y \in L, \langle y, e_i \rangle \geq 1, \langle y, e_j \rangle \geq 0 \text{ for every } j \leq m \right\}. \end{aligned}$$

Assume that for every $i \leq m$ the latter infimum attains on $y_i \in L$. Denoting $y_{ij} := \langle y_i, e_j \rangle$, we observe that $y_{ij} \geq 0$ and $y_{ii} \geq 1$ for every $i \leq m, j \leq m$, and that the matrix $\{y_{ij}\}$ has rank at most k . Since $m = d+k \geq 2k$, applying Lemma 3.1, we obtain

$$A = \sum_{i=1}^m \sum_{j=1}^m y_{ij} - m \geq \frac{m(m-1)}{2k-1} \geq \frac{m^2}{2k}.$$

This completes the proof. \square

4 A lower bound on the vertex index

In this section we apply our results to provide a sharp lower estimate for the vertex index of a centrally symmetric convex body and, in particular, to prove a conjecture of K. Bezdek on the lower bound for the illumination parameter of a centrally symmetric convex body.

The vertex index of a centrally symmetric (with respect to the origin) convex body \mathbf{K} , introduced in [BL], is defined as

$$\text{vein}(\mathbf{K}) = \inf \left\{ \sum_i \|p_i\|_{\mathbf{K}} \mid \mathbf{K} \subset \text{conv} \{p_i\}_i \right\}.$$

In other words, given $\mathbf{K} = -\mathbf{K}$ one looks for the convex polytope that contains \mathbf{K} and whose vertex set has the smallest possible closeness to 0 in the metric generated by \mathbf{K} .

The illumination parameter of a centrally symmetric convex body \mathbf{K} was introduced in [B1] as a cost function for the Boltyanski-Hadwiger illumination conjecture in the following way

$$\text{ill}(\mathbf{K}) = \inf \left\{ \sum_i \|p_i\|_{\mathbf{K}} \mid \{p_i\}_i \subset \mathbb{R}^d \text{ illuminates } \mathbf{K} \right\},$$

where “illuminates” means that for every q on the boundary of \mathbf{K} there exists a point p_i such that the ray starting at p_i and passing through q intersects the interior of \mathbf{K} (after the point q).

Let us note that both the vertex index and the illumination parameter, are affine invariants of \mathbf{K} ; that is, if $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is an invertible linear map then $\text{vein}(\mathbf{K}) = \text{vein}(T(\mathbf{K}))$ and $\text{ill}(\mathbf{K}) = \text{ill}(T(\mathbf{K}))$. It is also easy to see that $\text{vein}(\mathbf{K}) \leq \text{ill}(\mathbf{K})$. Thus both parameters are closely related and, as we mentioned in the introduction, they are closely connected to some important quantities and problems in asymptotic theory of normed spaces and in convex geometry.

In [B3] K. Bezdek conjectured (see Problem 6.3) that for every centrally symmetric d -dimensional body \mathbf{K} one has

$$\text{ill}(\mathbf{K}) \geq 2d.$$

The next theorem answers in affirmative the Bezdek’s conjecture and provides the lower bound on the vertex index of the centrally symmetric convex bodies.

Theorem 4.1 *Let \mathbf{K} be d -dimensional centrally symmetric (with respect to the origin) convex body. Then*

$$\text{ill}(\mathbf{K}) \geq \text{vein}(\mathbf{K}) \geq 2d.$$

Proof: As we mentioned above the left-hand side inequality is simple. We show the right-hand side inequality.

Let $\mathbf{K} \subset \mathbf{L} = \text{conv} \{p_i\}_{i \leq m}$. Without loss of generality we can assume that $\|p_i\|_{\mathbf{K}} \geq 1$ for every i . If $m \geq 2d$ then we trivially have

$$\sum_{i=1}^m \|p_i\|_{\mathbf{K}} \geq m \geq 2d.$$

Assume $m < 2d$. Since $\mathbf{K} = -\mathbf{K} \subset \mathbf{L}$, we have $\| -x \|_{\mathbf{L}} \leq \|x\|_{\mathbf{K}}$ for every $x \in \mathbb{R}^d$. Therefore, applying Theorem 3.2, we obtain

$$\sum_{i=1}^m \|p_i\|_{\mathbf{K}} \geq \sum_{i=1}^m \| -p_i \|_{\mathbf{L}} \geq 2d,$$

which completes the proof. □

Remark. Let \mathbf{K} be a cross-polytope in \mathbb{R}^d . Since \mathbf{K} has $2d$ vertices we have $\text{ill}(\mathbf{K}) \leq 2d$, which implies

$$\text{ill}(\mathbf{K}) = \text{vein}(\mathbf{K}) = 2d$$

(see [BL] for a direct proof of this equality). It shows that Theorem 4.1 is sharp.

References

- [B1] K. Bezdek, *Research problem 46*, Period. Math. Hungar. 24 (1992), 119–121.
- [B2] K. Bezdek, *Hadwiger-Levi's covering problem revisited*, J. Pach (ed.), New Trends in Discrete and Computational Geometry Berlin: Springer-Verlag. Algorithms Comb. 10, 199–233 (1993).
- [B3] K. Bezdek, *The illumination conjecture and its extensions*, Period. Math. Hungar., 53 (2006), 59–69.
- [BL] K. Bezdek, A. E. Litvak, *On the vertex index of convex bodies*, Adv. Math. 215 (2007), 626–641.

- [GL] E. D. Gluskin, A. E. Litvak, *On the asymmetry constant of a body with few vertices*, Geometriae Dedicata, 90 (2002), 45-48.
- [GoK] I. C. Gohberg, M. G. Krein, *Introduction to the theory of linear non-selfadjoint operators* (English). Translations of Mathematical Monographs. 18. Providence, RI: American Mathematical Society (AMS). XV, 1969.
- [Gr] B. Grünbaum, *Measures of symmetry for convex sets*, Proc. Sympos. Pure Math. 1963, Vol. VII pp. 233–270 Amer. Math. Soc., Providence, R.I.
- [MS] H. Martini, V. Soltan, *Combinatorial problems on the illumination of convex bodies*, Aequationes Math. 57 (1999), 121-152.
- [S] W. Stinespring, *A sufficient condition for an integral operator to have a trace*, J. Reine Angew. Math. 200 (1958), 200-207.
- [W] H. Weyl, *Inequalities between the two kinds of eigenvalues of a linear transformation*, Proc. Natl. Acad. Sci. USA 35 (1949), 408-411.

A. E. Litvak, Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB, Canada, T6G 2G1.

e-mail: alexandr@math.ualberta.ca

E.D. Gluskin, School of Mathematical Sciences, Tel Aviv University, Ramat Aviv, Tel Aviv 69978, Israel.

e-mail: gluskin@post.tau.ac.il