An example of a convex body without symmetric projections.

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Abstract

Many crucial results of the asymptotic theory of symmetric convex bodies were extended to the non-symmetric case in recent years. That led to the conjecture that for every n-dimensional convex body K there exists a projection P of rank k, proportional to n, such that PK is almost symmetric. We prove that the conjecture does not hold. More precisely, we construct an n-dimensional convex body K such that for every $k > C\sqrt{n \ln n}$ and every projection P of rank k, the body PK is very far from being symmetric. In particular, our example shows that one can not expect a formal argument extending the "symmetric" theory to the general case.

0 Introduction

The asymptotic theory of finite dimensional normed spaces studies the behavior of various functionals on the set of all finite dimensional normed spaces, or equivalently, the behavior of centrally-symmetric convex bodies in $\mathbb{R}^n$, when the dimension grows to infinity. The development of the theory in the last years required to abandon the condition of central symmetry. It turns out that many results of the theory remain valid in the non-symmetric case, although their proofs require new and different arguments (see e.g., [GGM], [LMP], [MP], [R1], [R2], [LT] and references therein). Unexpected similarity of results concerning general convex bodies and their counterparts in the centrally symmetric case lead a number of authors to the following conjecture which we state a bit vaguely.

*This author holds Lady Davis Fellowship.

Conjecture. *For every convex body K in $\mathbb{R}^n$ there exists an orthogonal projection P of rank k, proportional to n, such that PK is almost symmetric.*

If this conjecture were true, an extension of large class of problems in the asymptotic theory of finite dimensional normed spaces to the non-symmetric case would be automatic. However, it is not so, and in this note we construct a counterexample. Our ideas and intuitions have two sources. The first are constructions of random symmetric convex bodies that have been initiated in [G1] (an exhaustive survey of this direction can be found in [MT]). In particular, the main line of the present argument is close to [G2]. The second source is a recent paper [LT] where it was shown, among other results, that a random projection of a simplex is very far from being symmetric. More precisely, if P is a random orthogonal projection of rank $k > \ln n$ and $S \subset \mathbb{R}^n$ is the regular simplex then the distance from PS to any centrally symmetric body is, with high probability, larger than $c\sqrt{k/\ln n}$, where c is an absolute constant. This shows that the conjecture fails for random projections. It should be noted, however, that there exist projections of the simplex of proportional dimension which are close to symmetric ([GGM]). We will show in the present note that there exists a convex body $K \subset \mathbb{R}^n$ such that its image under any rank k projection, with $k > \sqrt{n \ln n}$, has the distance from any centrally symmetric body larger than $ck/\sqrt{n \ln n}$.

1 Definitions, notations, known results.

We will use the standard notation from the local theory of Banach spaces (see e.g. [MS]). Given a finite set $\mathcal{N}$, its cardinality is denoted by $|\mathcal{N}|$. Denote the canonical Euclidean norm on $\mathbb{R}^n$ by $|\cdot|$, the Euclidean unit ball by B_2^n, and the Euclidean unit sphere by S^{n-1}. The distance between a point $a \in \mathbb{R}^n$ and a set $K \subset \mathbb{R}^n$ is

$$\text{dist } (a, K) = \min \{|a - x| \mid x \in K\}.$$

By a convex body $K \subset \mathbb{R}^n$ we shall always mean a compact convex set with the non-empty interior, and without loss of generality we shall assume that $0 \in \text{Int } K$. The gauge of K is denoted by $\|\cdot\|_K$, i.e., $\|x\|_K = \inf \{\lambda > 0 \mid x \in \lambda K\}$.

Given convex bodies K, L in $\mathbb{R}^n$, we define the geometric distance by

$$\tilde{d}(K, L) = \inf\{\alpha\, \beta \mid \alpha > 0, \beta > 0, (1/\beta)L \subset K \subset \alpha L\}.$$

The Banach–Mazur distance is defined by

$$d(K,L) = \inf\left\{\tilde{d}(u(K-z), L-x)\right\},$$

where infimum is taken over all $z,\ x \in I\!R^n$ and all linear operators $u : I\!R^n \to I\!R^n$. Clearly, if K and L are centrally symmetric, then the infimum is attained at $z = x = 0$. Thus, in the centrally symmetric case the definition coincides with the standard definition of the Banach-Mazur distance between balls of normed spaces. Moreover, if L is centrally symmetric then

$$d(K,L) \le 2\inf\left\{\tilde{d}\,(u(K-z), L)\right\}.$$

For simplicity we will use $d(K,L)$ for $\inf\left\{\tilde{d}\,(u(K-z), L)\right\}$ in the case $L = -L$.

Denote by $\mathcal{C}^n$ the set of all centrally symmetric bodies in $I\!R^n$. Let K be a convex body in $I\!R^n$. The quantity

$$d(K, \mathcal{C}^n) = \inf\{d(K,B) \mid B \in \mathcal{C}^n\}$$

measures, in a sense, the "non-symmetry" of K. (See [Gr], where the problem of the measure of "non-symmetry" is discussed.) By compactness, there exist $a \in K$ and a centrally symmetric convex body $B \subset I\!R^n$ such that $d(K, \mathcal{C}^n) = \tilde{d}(K-a, B)$. Observe that we also have

$$\begin{aligned} d(K, \mathcal{C}^n) &= \tilde{d}(K-a, (K-a) \cap -(K-a)) \\ &= \tilde{d}(K-a, \operatorname{conv}\{(K-a) \cup -(K-a)\}). \end{aligned}$$

That is, $(K-a)\cap -(K-a)$ and $\operatorname{conv}\{(K-a)\cup -(K-a)\}$ are two centrally symmetric bodies closest to K. Also note that for any $a \in K$ we have

$$\frac{1}{2}(K-K) \subset \operatorname{conv}\{(K-a) \cup -(K-a)\} \subset K-K.$$

Given a convex body $K \subset I\!R^n$ and $A \ge 1$, we say that K is A-symmetric if $d(K, \mathcal{C}^n) \le A$, and that K is A-symmetric with respect to a center $a \in I\!R^n$ if $\tilde{d}(K-a, B) \le A$ for some centrally symmetric body B. We also denote $K-a$ by K_a. By the above remark, the body K is A-symmetric with respect to a center $a \in I\!R^n$ if and only if $\|-x\|_{K_a} \le A\|x\|_{K_a}$ for every $x \in I\!R^n$.

For a linear operator $u : I\!R^n \to I\!R^n$, by $\|u\|$ we denote its operator norm $\|u : \ell_2^n \to \ell_2^n\|$. Finally, the letters C, c, c_0, c_1, ... denote absolute constants whose values may be different from line to line.

We shall now describe our probabilistic setting and the background results. Let us note at the beginning that we shall work with random vectors uniformly distributed on the sphere; however, similar calculations could be done with random Gaussian vectors, similarly as e.g., in [MT].

As usual, $\mathbf{E}$ denotes the expectation and $\mathbf{Pr}$ denotes the probability. Let ω be a random vector uniformly distributed on S^{n-1} and P be an orthogonal projection of rank $1 \leq k < n$. The following well-known inequalities easily follow from the concentration of measure phenomena (see e.g., [MS]). There is an absolute constant c such that

$$\mathbf{Pr}\left(\left\{\frac{\sqrt{k}}{2\sqrt{n}} \leq |P\omega| \leq \frac{2\sqrt{k}}{\sqrt{n}}\right\}\right) \geq 1 - e^{-ck}, \tag{1}$$

and for every $t > 0$,

$$\mathbf{Pr}\left(\left\{|P\omega| \geq (1+t)\sqrt{k/n}\right\}\right) \leq e^{-ckt^2}. \tag{2}$$

The direct computations show that for every $p > 1$ and for every $x \in B_2^n$

$$(\mathbf{E}\,|\langle \omega, x\rangle|^p)^{1/p} \leq c\sqrt{p/n},$$

where c is an absolute constant. It follows then that for every integer m and every $\{x_i\}_{i\leq m} \subset B_2^n$ we have

$$\begin{aligned}\mathbf{E}\max_{i\leq m}|\langle \omega, x_i\rangle| &\leq \mathbf{E}\left(\sum_{i=1}^m |\langle \omega, x_i\rangle|^p\right)^{1/p} \\ &\leq \left(\mathbf{E}\sum_{i=1}^m |\langle \omega, x_i\rangle|^p\right)^{1/p} \leq cm^{1/p}\sqrt{p/n}.\end{aligned}$$

Choosing $p = \ln m$ we obtain that there is an absolute constant c_0 such that

$$\mathbf{E}\max_{i\leq m}|\langle \omega, x_i\rangle| \leq c_0\sqrt{\frac{\ln m}{n}}. \tag{3}$$

for every every $\{x_i\}_{i\leq m} \subset B_2^n$. Of course the same estimate holds for any random vector ω taking values in B_2^n, whose distribution is invariant under rotations.

Let $(\Omega, \mathcal{F}, \mathbf{Pr})$ be a probability space. Let $\emptyset \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset ... \subset \mathcal{F}$ be σ-fields. Let f_0, f_1, f_2, ... be a sequence of random variables such that f_i is

measurable with respect to $\mathcal{F}_i$ for every i, and the conditional expectation $\mathbf{E}(f_i \mid \mathcal{F}_{i-1}) = f_{i-1}$. Then the sequence $\{f_i\}$ is called a martingale with respect to $\{\mathcal{F}_i\}$.

We shall use the following well-known estimate for large deviation of martingales with bounded martingale differences. (As usual $\|d\|_\infty$ denotes the essential supremum of the absolute value of a random variable d).

Lemma 1.1 ([A]) *Let $0 = f_0$, f_1, ..., f_m be a martingale and let $d_i = f_i - f_{i-1}$ be bounded for every $i \le m$. Then for every positive λ one has*

$$\mathbf{Pr}\left(\left\{\sum_{i=1}^{m} d_i \ge \lambda\right\}\right) \le e^{-\lambda^2/(2M)}$$

where $M = \sum_{i=1}^{m} \|d_i\|_\infty^2$.

Let h_1, ..., h_m be independent identically distributed random variables with $\mathbf{E}h_i = a$ (where $a > 0$) and $\|h_i\|_\infty \le b$. Applying Lemma 1.1 to $f_i = \sum_{j=1}^{i}(a - h_i)$ and $\lambda = ma/2$ we get

$$\mathbf{Pr}\left(\left\{\sum_{i=1}^{m} h_i \le ma/2\right\}\right) \le e^{-ma^2/(32b^2)} \tag{4}$$

Finally we need the estimates for cardinality of nets in Banach spaces. Recall that a set $\mathcal{N}$ is called an ε-net for the set K with respect to given norm $\|\cdot\|$ if for every $x \in K$ there is $a \in \mathcal{N}$ such that $\|x - a\| \le \varepsilon$. The following fact gives a standard estimate for cardinality of nets (see e.g., [MT], Lemma 6).

Fact 1.2 *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ and B be its unit ball. Then for every $0 < \varepsilon \le 1$ there is $\mathcal{N}$, an ε-net for the B with respect to the norm $\|\cdot\|$, such that $|\mathcal{N}| \le (3/\varepsilon)^n$.*

Remark. It is well known and easy to check that if $\mathcal{N}$ is an ε-net in B in the norm $\|\cdot\|$, then for any subset $K \subset B$ there exists an (2ε)-net $\mathcal{N}_0 \subset K$ with cardinality $|\mathcal{N}| = |\mathcal{N}_0|$. Consider the case when B is the set of all linear operators $u : \mathbb{R}^n \to \mathbb{R}^n$ satisfying $\|u : \ell_2^n \to \ell_2^n\| \le 1$, and K is the set of all orthogonal projections on $\mathbb{R}^n$ of rank k (for a fixed $1 \le k \le n$). Thus there exists an ε-net $\mathcal{N}$ consisting of orthogonal projections of rank k and such that $|\mathcal{N}| \le (6/\varepsilon)^{n^2}$. This is a very rough estimate, however it is sufficient for our purposes in this paper. We refer the interested reader to [Sz] where the investigation of ε-nets for the set of all rank k orthogonal projections on $\mathbb{R}^n$ was done in detail.

2 An example

Theorem 2.1 *Let n be a positive integer. There exist a convex body $K \subset \mathbb{R}^n$ such that for every projection P the distance of PK from the set of centrally symmetric convex bodies is bounded from below by*

$$d(PK, \mathcal{C}^k) \ \geq c\frac{k}{\sqrt{n \ln n}},$$

where $k = \text{rank } P$ and $c > 0$ is an absolute constant.

Remark. Since clearly $d(PK, \mathcal{C}^k) \geq 1$, the theorem is of interest for $k \geq \sqrt{n \ln n}/c$ only.

Lemma 2.2 *Let $k \leq m$ be positive integers. Let v_1,...,v_m be independent identically distributed random vectors in $\mathbb{R}^k$ with rotation invariant distribution. Let w_1,...,w_m be random vectors in $\mathbb{R}^k$ such that v_i is independent of $\{w_1, ..., w_{i-1}\}$ for every $i \leq m$. Assume that v_i's are distributed in RB_2^k and w_i's are distributed in rB_2^k for some R, $r > 0$. Then*

$$\mathbf{Pr}\left(\left\{\sum_{i=1}^m \max_{j<i} |\langle v_i, w_j\rangle| \geq Rr\left(\lambda + c_0 m\sqrt{(\ln m)/k}\right)\right\}\right) \ \leq e^{-\lambda^2/(8m)},$$

where c_0 is the constant from (3).

Proof: Denote $f_i = \max_{j<i} |\langle v_i, w_j\rangle|$. By conditions of the lemma we have $|\langle v_i, w_j\rangle| \leq Rr$. Let $\mathcal{F}_i$ be the σ-algebra generated by v_1, ..., v_i, w_1, ..., w_i. Let $\bar{f}_i$ be the conditional expectation $\mathbf{E}(f_i|\mathcal{F}_{i-1})$ and $d_i = f_i - \bar{f}_i$. Then $u_i = \sum_{j=1}^i d_j$, $i \leq m$, is a martingale with respect to $\{\mathcal{F}_i\}$. Since $\|d_i\|_\infty \leq 2Rr$, $i \leq m$, by Lemma 1.1 we obtain that

$$\mathbf{Pr}\left(\{u_m \geq Rr\lambda\}\right) \leq e^{-\lambda^2/(8m)}.$$

By (3), we have $\bar{f}_i \leq c_0 Rr\sqrt{(\ln m)/k}$, for $i \leq m$. Thus

$$\begin{aligned}
&\mathbf{Pr}\left(\left\{\sum_{i=1}^m \max_{j<i} |\langle v_i, w_j\rangle| \geq Rr\left(c_0 m\sqrt{(\ln m)/k} + \lambda\right)\right\}\right) \\
&= \mathbf{Pr}\left(\left\{u_m + \sum_{i=1}^m \bar{f}_i \geq Rr\left(c_0 m\sqrt{(\ln m)/k} + \lambda\right)\right\}\right) \\
&\leq \mathbf{Pr}\left(\{u_m \geq Rr\lambda\}\right) + \mathbf{Pr}\left(\left\{\sum_{i=1}^m \bar{f}_i > Rrc_0 m\sqrt{(\ln m)/k}\right\}\right) \\
&\leq e^{-\lambda^2/(8m)}.
\end{aligned}$$

This proves the lemma. $\square$

Remark. In the sequel we shall use the lemma as follows. Let h_1, ..., h_m be independent identically distributed random vectors in $\mathbb{R}^k$ with rotation invariant distribution. Let $x \in \mathbb{R}^k$. Assume that h_i's are distributed in rB_2^k. Applying the lemma for $w_i = h_i - x$ and $v_i = \chi_{RB_2^k} \cdot h_i$, where χ_K denotes the indicator of the set K, we obtain

$$\mathbf{Pr}\left(\left\{\sum_{i=1}^{m} \max_{j<i} \left|\langle \chi_{RB_2^k} h_i, h_j - x\rangle\right| \geq 2R(r+|x|)\lambda\right\}\right) \leq e^{-\lambda^2/(8m)} \qquad (5)$$

for every $\lambda \geq c_0 m\sqrt{(\ln m)/k}$.

Lemma 2.3 *Let $k \leq n \leq m$ be positive integers, $y \in 2B_2^n$. Let h_1,...,h_m be independent random vectors uniformly distributed on S^{n-1}. Let P be an orthogonal projection of rank k. Denote $v_i = Ph_i$, $i \leq m$, and $x = Py$. Then there are absolute constants c, c_1 such that for every $1 \leq A \leq c_1 k/\sqrt{n \ln m}$ we have*

$$\mathbf{Pr}\left(\left\{v_i \in A \operatorname{absconv}\{v_j - x\}_{j\neq i} + \frac{\sqrt{k}}{16\sqrt{n}} PB_2^n, \text{ for all } i\right\}\right) \leq 2e^{-cmk/(nA^2)}$$

Proof: Denote $\bar{v}_i = v_i \chi_B$, $i \leq m$, where $B = 2\sqrt{k/n} PB_2^n$. Clearly, if $v_i \in A \operatorname{absconv}\{v_j - x\}_{j\neq i} + \frac{\sqrt{k}}{16\sqrt{n}} PB_2^n$ then $\bar{v}_i \in A \operatorname{absconv}\{v_j - x\}_{j\neq i} + \frac{\sqrt{k}}{16\sqrt{n}} PB_2^n$, which means that there exist $\lambda_j \in [-1,1]$ with $\sum |\lambda_j| \leq 1$ and $z \in \frac{\sqrt{k}}{16\sqrt{n}} PB_2^n$ such that

$$\bar{v}_i = A\sum_{j\neq i} \lambda_j (v_j - x) + z.$$

Taking scalar product with $\bar{v}_i$ we obtain

$$\langle \bar{v}_i, \bar{v}_i\rangle \leq A\sum_{j\neq i} |\langle \bar{v}_i, \lambda_j (v_j - x)\rangle| + |\langle \bar{v}_i, z\rangle| \leq A \max_{j\neq i} |\langle \bar{v}_i, v_j - x\rangle| + \frac{k}{8n}.$$

Therefore if $v_i \in A \operatorname{absconv}\{v_j - x\}_{j\neq i} + \frac{\sqrt{k}}{16\sqrt{n}} PB_2^n$ for every i then $|\bar{v}_i|^2 - \frac{k}{8n} \leq A \max_{j\neq i} |\langle \bar{v}_i, v_j - x\rangle|$ for every i, which implies

$$\sum_{i=1}^{m} |\bar{v}_i|^2 - \frac{km}{8n} \leq A\sum_{i=1}^{m} \max_{j\neq i} |\langle \bar{v}_i, v_j - x\rangle|.$$

The probability of the last event is less than or equal to

$$\mathbf{Pr}\left(\left\{\sum_{i=1}^{m}|\bar{v}_i|^2 \le \frac{km}{4n}\right\}\right) + \mathbf{Pr}\left(\left\{\frac{km}{8nA} \le \sum_{i\le m}\max_{j<i}|\langle \bar{v}_i, v_j - x\rangle| + \sum_{i\le m}\max_{j>i}|\langle \bar{v}_i, v_j - x\rangle|\right\}\right). \quad (6)$$

Clearly $\mathbf{E}\,|v_i|^2 = \frac{k}{n}$. Thus, using (2), one can see that $\mathbf{E}\,|\bar{v}_i|^2 \ge \frac{k}{2n}$. Since $\bar{v}_i \in 2\sqrt{k/n}B_2^n$, we have also that $\| \, |\bar{v}_i|^2 \, \|_\infty \le 4k/n$. Applying (4) we obtain

$$\mathbf{Pr}\left(\left\{\sum_{i=1}^{m}|\bar{v}_i|^2 \le \frac{km}{4n}\right\}\right) \le e^{-cm}$$

for an absolute constant c. The second term in (6) is bounded by $e^{-c_1 mk/(A^2 n)}$ because of estimate (5) with $r = 1$, $R = 2\sqrt{k/n}$, $\lambda = \frac{\sqrt{k}m}{96\sqrt{n}A}$. That proves the lemma. □

Proof of Theorem 2.1: Let us note that it is enough to prove the Theorem for orthogonal projections only. Indeed, let P be any projection. Denote by Q the orthogonal projection with the same kernel. Clearly, Q has the same rank as P, $QP = Q$ and thus if PK is A-symmetric then QK is A-symmetric as well.

Let $m \in \mathbb{N}$, $m = [c_0 n^3]$, and $\delta = c/\sqrt{n}$, where c_0 and c are absolute constants, which will be chosen later.

Given $k \le n$ let $A_k > 1$ be parameter, which will be specified later ($A_k \approx k/\sqrt{n \ln n}$).

Let $\mathcal{N}_k$ be a δ-net of projections of rank k. Let $\mathcal{M}$ be a δ-net in B_2^n. By Fact 1.2 and the remark that follows, we can assume that $|\mathcal{N}_k| \le (6/\delta)^{n^2}$ and $|\mathcal{M}| \le (3/\delta)^n$.

First we show that there exist vectors h_1, h_2, ..., h_m in $\mathbb{R}^n$ such that for every $k \le n$, satisfying $A_k > 1$, for every $Q \in \mathcal{N}_k$ and every $b \in \mathcal{M}$ the following condition does not hold at least for one $i \in \{1, 2, ..., m\}$

$$-Qh_i \in A_k \mathrm{conv}\{Q(h_j - x_b)\}_{j\ne i} + 2(A_k + 1)\delta Q B_2^n, \quad (7)$$

where $x_b = (1 + 1/A_k)Qb \in 2B_2^n$.

Indeed, consider independent random vectors h_1, ..., h_m uniformly distributed on S^{n-1}. Then by Lemma 2.3 the probability that (7) holds for every i with fixed Q and b is less than

$$\mathbf{Pr}\left(\{(7) \text{ holds } \forall i\}\right) \le 2e^{-c_1 mk/(nA_k^2)}$$

for $A_k \le c_2 k/\sqrt{n \ln m}$, where c_1, c_2 are absolute constants. Take $A_k = c_2 k/\sqrt{n \ln m}$. Then $2(A_k+1)\delta \le \sqrt{k}/(16\sqrt{n})$ for an appropriate choice of an absolute constant c. Therefore probability that there exist $Q \in \mathcal{N}_k$, $b \in \mathcal{M}$ such that the inclusion (7) holds for every i is less than

$$\begin{aligned}
&\sum_{Q\in\mathcal{N}_k}\sum_{b\in\mathcal{M}} \mathbf{Pr}\left(\left\{\forall i \ : \ Qh_i \in A_k \operatorname{absconv}\left\{Qh_j - x_b\right\}_{j\neq i} + \frac{\sqrt{k}}{16\sqrt{n}} QB_2^n\right\}\right) \\
&\quad \le \ 2\left(\frac{6}{\delta}\right)^{n^2}\left(\frac{3}{\delta}\right)^{n} e^{-c_1 mk/(nA_k^2)} \\
&\quad \le \ 2\exp\left(2n^2 \ln\left(6\sqrt{n}/c\right) - c_1 m \ln(m)/(c_2^2 k)\right).
\end{aligned}$$

Thus there are absolute constants c_0, c_3, c_4 such that for $m = [c_0 n^3]$ the probability

$$\begin{aligned}
&\mathbf{Pr}\left(\{\exists k \ \exists Q \in \mathcal{N}_k, \exists b \in \mathcal{M} \text{ such that (7) holds } \forall i\}\right) \\
&\le \ \sum_k \mathbf{Pr}\left(\{\exists Q \in \mathcal{N}_k, \exists b \in \mathcal{M} \text{ such that (7) holds } \forall i\}\right) \\
&\le \ 2n\exp\left(-c_3 n^2 \ln n\right) \le \exp\left(-c_4 n^2 \ln n\right).
\end{aligned}$$

Hence we can find vectors h_1, h_2, ..., h_m with the desired property. Moreover, the random choice gives such vectors with probability larger than $1 - \exp\left(-c_4 n^2 \ln n\right)$.

Now take such vectors and set $B = \operatorname{conv}\{h_i\}_{i\le m}$. Fix k. Let P be an orthogonal projection of rank k, $A = A_k$. Then the body PB is A-symmetric with respect to some center a if and only if $-Ph_i + a \in A(\operatorname{conv}\{h_j\}_{j\le m} - a)$ for every $i \le m$. The last inclusion implies

$$-Ph_i \in A\operatorname{conv}\{Ph_j\}_{j\neq i} - (A+1)a$$

for every i. By the definition of δ-net there are $Q \in \mathcal{N}_k$ and $b \in \mathcal{M}$ such that $\|Q - P\| \le \delta$ and $|a - Qb| \le \delta$. Therefore $\max_i |(Q-P)h_i| \le \delta$ and

$$\begin{aligned}
&\operatorname{dist}\left(-Qh_i, A\operatorname{conv}\left\{Qh_j\right\}_{j\neq i} - (A+1)\,Qb\right) \\
&\le \ \delta + A\delta + (A+1)\,\delta = 2\,(A+1)\,\delta.
\end{aligned}$$

This means that for some $Q \in \mathcal{N}_k$ and $b \in \mathcal{M}$ the inclusion (7) holds for every i. That contradicts with the choice of $\{h_i\}$. Thus we obtain that PB is not A-symmetric for $A = c_2 k/\sqrt{n \ln m}$. This proves the theorem. $\square$

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