On the minimum of several random variables *

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Abstract

For a given sequence of real numbers $a_1, \ldots, a_n$ we denote the k-th smallest one by $k\text{-}\min_{1\le i\le n} a_i$. Let $\mathcal{A}$ be a class of random variables satisfying certain distribution conditions (the class contains $N(0,1)$ Gaussian random variables). We show that there exist two absolute positive constants c and C such that for every sequence of real numbers $0 < x_1 \le \ldots \le x_n$ and every $k \le n$ one has

$$c \max_{1\le j\le k} \frac{k+1-j}{\sum_{i=j}^n 1/x_i} \le \mathbb{E}\, k\text{-}\min_{1\le i\le n} |x_i \xi_i| \le C \ln(k+1) \max_{1\le j\le k} \frac{k+1-j}{\sum_{i=j}^n 1/x_i},$$

where $\xi_1, \ldots, \xi_n$ are independent random variables from the class $\mathcal{A}$. Moreover, if $k=1$ then the left hand side estimate does not require independence of the ξ_i's. We provide similar estimates for the moments of $k\text{-}\min_{1\le i\le n} |x_i\xi_i|$ as well.

1 Introduction

For a given sequence of real numbers $a_1, \ldots, a_n$ we denote the k-th smallest one by $k\text{-}\min_{1\le i\le n} a_i$; thus, $1\text{-}\min_{1\le i\le n} a_i = \min_{1\le i\le n} a_i$, and $2\text{-}\min_{1\le i\le n} a_i$ is the next smallest, etc., and $(k\text{-}\min_{1\le i\le n} a_i)_{k=1}^n$ is the non-decreasing rearrangement of the sequence $(a_i)_{i=1}^n$. In the same way we denote the k-th biggest number by $k\text{-}\max_{1\le i\le n} a_i$.

In the paper [GLSW1] we considered expressions of the form

$$\mathbb{E} \sum_{k=1}^m k\text{-}\max_{1\le i\le n} |x_i f_i|^p,$$

where $f_1, f_2, \ldots, f_n$ are random variables and $x_1, x_2, \ldots, x_n$ are real numbers. Since the functions $(\sum_{k=1}^m k\text{-}\max_{1\le i\le n} |x_i|^p)^{1/p}$ are norms on $\mathbb{R}^n$, such forms appear naturally in the study of various parameters associated with the geometry of Banach spaces [GLSW2]. Other applications of these forms can be found in [KS1] and [KS2].

The striking difference in the present study is that we now consider expressions of the form $\left(\sum_{k\in I} k\text{-}\min_{1\le i\le n} |x_i|^p\right)^{1/p}$ for subsets $I \subseteq \{1, \ldots, n\}$. These are not norms if I is not an integer interval starting at 1. Hence, for a given sequence of random variables $f_1, \ldots, f_n$, the computation of expressions such as

$$\mathbb{E}\, k\text{-}\min_{1\le i\le n} |f_i|^p = \mathbb{E}\,(n-k+1)\text{-}\max_{1\le i\le n} |f_i|^p$$

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requires completely different techniques. Such minima, also called *order statistics*, have been intensively studied during last century. We refer an interested reader to [AB] and [DN] for basic facts, known results, and references. Most works dealt with the case of independent identically distributed random variables. Sometimes the condition "to be identically distributed" was substituted by the condition"the f_i's have the same first and the same second moments". In this paper we drop these conditions and deal with sequences of random variables having no restrictions on their moments. Applications of the current paper are related to important electrical engineering problems on the minimization of the data loss of signals emanating from electrical networks. Applications appear also in the study of the multifold K-functional or its geometric equivalent, the norm with unit ball $B = \mathrm{co}(\cup_{i=1}^m B_i)$, where the B_i's are unit balls of symmetric normed spaces.

Now we describe our setting and results. Let $\alpha > 0$, $\beta > 0$ be parameters. We say that a random variable ξ satisfies the (α, β)-condition if

$$\mathbb{P}\left(|\xi| \le t\right) \le \alpha t \qquad \text{for every} \qquad t \ge 0 \tag{1}$$

and

$$\mathbb{P}\left(|\xi| > t\right) \le e^{-\beta t} \qquad \text{for every} \qquad t \ge 0. \tag{2}$$

Note that this forces the condition $\alpha\, t + e^{-\beta t} \ge 1$ for all $t \ge 0$, which implies $\alpha \ge \beta$.

Below (Claim 1 and the remark following it) we will see that many random variables, including $N(0,1)$ Gaussian variables (with $\alpha = \beta = \sqrt{2/\pi}$) and exponentially distributed variables (with $\alpha = \beta = 1$), satisfy this condition. We study first the moments of the minimum of a sequence of such random variables. Let $x_1, \ldots, x_n$ be a sequence of real numbers and $\xi_1, \ldots, \xi_n$ be independent random variables satisfying the (α, β)-condition. We obtain that for every $p > 0$ one has

$$\tfrac{1}{1+p}\, \alpha^{-p} \left(\sum_{i=1}^n \frac{1}{|x_i|} \right)^{-p} \le \mathbb{E} \min_{1 \le i \le n} |x_i \xi_i|^p \le \beta^{-p}\, \Gamma(1+p) \left(\sum_{i=1}^n \frac{1}{|x_i|} \right)^{-p},$$

where $\Gamma(\cdot)$ is the Gamma-function. Moreover, the left hand side estimate does not require the independence of the ξ_i's. In particular, it implies that for every $p > 0$

$$\mathbb{E} \min_{1 \le i \le n} |x_i g_i|^p \le \Gamma(2+p)\ \mathbb{E} \min_{1 \le i \le n} |x_i f_i|^p,$$

where $g_1, \ldots, g_n$ are independent $N(0,1)$ Gaussian random variables and $f_1, \ldots, f_n$ are $N(0,1)$ Gaussian random variables (not necessarily independent). Taking $p = 1$, we have

$$\mathbb{E} \min_{1 \le i \le n} |x_i g_i| \le 2\ \mathbb{E} \min_{1 \le i \le n} |x_i f_i|.$$

This result should be compared with well known Šidák's inequality ([S], [G]), saying that

$$\mathbb{E} \max_{1 \le i \le n} |x_i g_i| \ge \mathbb{E} \max_{1 \le i \le n} |x_i f_i|.$$

In other words our result is, in a sense, an *inverse Šidák's inequality.* It would be nice to eliminate the factor 2 from it.

We generalize our estimates for the expectation of the minimum to the case of the k-th minimum. For $0 < x_1 \le x_2 \le \ldots \le x_n$ and independent random variables $\xi_1, \ldots, \xi_n$ satisfying the (α, β)-condition, we obtain that there are two absolute positive constants c and C such that for every $p > 0$ one has

$$c_p\, \alpha^{-1} \max_{1 \le j \le k} \frac{k+1-j}{\sum_{i=j}^n 1/x_i} \le \left(\mathbb{E}\ k\text{-}\min_{1 \le i \le n} |x_i \xi_i|^p \right)^{1/p} \le C(p,k)\, \beta^{-1} \max_{1 \le j \le k} \frac{k+1-j}{\sum_{i=j}^n 1/x_i},$$

where $c_p = c^{1+1/p}$ and $C(p,k) = C\max\{p, \ln(k+1)\}$. We would like to emphasize that the estimates are pretty sharp, in particular the ratio of upper and lower bounds surprisingly does not depend on n and, up to a constant depending only on p, is bounded by $\ln(k+1)$.

The latter result implies that we may evaluate sums of the form

$$\sum_{k\in I} \mathbb{E}\; k\text{-}\min_{1\le i\le n} |x_i g_i|^p,$$

where $I \subset \{1,2,...,n\}$ is any subset of integers and the g_i's are $N(0,1)$ Gaussian random variables.

The paper is organized as follows: in the next section, Section 2, we introduce the notations, quote some known facts, and prove that random variables with certain densities, including Gaussian and exponential, satisfy the (α,β)-condition. In Section 3, we provide combinatorial results used in the proofs. In Section 4, we prove our main theorems. Finally, in Section 5, we provide some examples.

An extended abstract of this work appeared in [GLSW3].

2 Notation and preliminaries

Given $A \subset \mathbb{N}$ we denote its cardinality by $|A|$. Given a set E we denote its complement by E^c. We say that $(A_j)_{j=1}^k$ is a partition of $\{1,2,\dots,n\}$ if $\emptyset \neq A_j \subset \{1,2,\dots,n\}$, $j\le k$, $\cup_{j\le k} A_j = \{1,2,\dots,n\}$, and $A_i \cap A_j = \emptyset$ for $i \neq j$. The canonical Euclidean norm and the canonical inner product on $\mathbb{R}^n$ we denote by $|\cdot|$ and $\langle\cdot,\cdot\rangle$. By $1/t$ we mean ∞ if $t=0$ and 0 if $t=\infty$.

As we defined above, $(k\text{-}\min_{1\le i\le n} a_i)_{k=1}^n$ denotes the non-decreasing rearrangement of the sequence $(a_i)_{i=1}^n$, i.e. $k\text{-}\min(a_i)_{i=1}^n$ is the k-th smallest element of the sequence.

We will use the following simple properties of $k\text{-}\min$ which hold for every sequence $(a_i)_{i=1}^n$.

For every $r < k$

$$k\text{-}\min(a_i)_{i=1}^n \ge (k-r)\text{-}\min(a_i)_{i=r+1}^n. \tag{3}$$

For every partition $(A_j)_{j\le k}$ of $\{1,2,...,n\}$

$$k\text{-}\min(a_i)_{i=1}^n \le \max_{j\le k}\left\{\min_{i\in A_j} a_i\right\}_{j\le k}. \tag{4}$$

Now we recall some definitions and estimates connected to the Gaussian distribution.

Let x and y be non-negative real numbers. The Gamma-function is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$$

Note that

$$x\Gamma(x) = \Gamma(1+x), \qquad \Gamma(n+1) = n!,$$

and, by Stirling's formula, for every $x \ge 1$

$$\sqrt{2\pi x}\;\left(\frac{x}{e}\right)^x < \Gamma(x+1) < \sqrt{2\pi x}\;\left(\frac{x}{e}\right)^x e^{\frac{1}{12x}}. \tag{5}$$

Finally, we show examples of random variables, satisfying the (α,β)-condition.

Claim 1 *Let $q \ge 1$. Let ξ be a non-negative random variable with the density function $p(s) = c_q \exp(-s^q)$, where $c_q = 1/\Gamma(1+1/q)$. Then ξ satisfies (1) and (2) with parameters $\alpha = \beta = c_q$.*

Remark. An important case is the case $q = 2$ which corresponds to the Gaussian random variable. Claim 1 implies that $N(0,1)$ Gaussian random variables satisfy the (α, β)-condition with $\alpha = \beta = \sqrt{2/\pi}$. We would like also to note that if $q = 1$ then we have an exponentially distributed random variable. In this case $\alpha = \beta = 1$.

Proof of Claim 1. The case $q = 1$ is trivial. So we assume that $q > 1$. Clearly we have

$$\mathbb{P}\left(|\xi| \le t\right) = c_q \int_0^t e^{-s^q} ds \le c_q t,$$

which shows $\alpha = c_q$.

Now consider the function g defined on $[0, \infty)$ by

$$g(x) = \exp\left(-c_q\, x\right) - c_q \int_x^\infty e^{-s^q}\, ds.$$

Then $g(0) = \lim_{x\to\infty} g(x) = 0$ and $g'(x) = c_q \left(\exp\left(-x^q\right) - \exp\left(-c_q\, x\right)\right)$. Hence, $g'(x) \ge 0$ on $[0, c_q^{1/(q-1)}]$ and $g'(x) \le 0$ for $x \ge c_q^{1/(q-1)}$. It shows that $g(x) \ge 0$ for every $x \ge 0$. Therefore

$$\mathbb{P}\left(|\xi| > t\right) = c_q \int_t^\infty e^{-s^q} ds \le e^{-c_q t},$$

i.e. $\beta = c_q$. □

3 Combinatorial results

In this section we prove some combinatorial results, which will be used later in the proofs of theorems.

First we quote the following result on symmetric means ([HLP]).

Lemma 2 *Let $1 \le l \le n$. Let a_i, $i = 1, \dots, n$ be nonnegative real numbers. Then*

$$\sum_{\substack{A \subset \{1,2,\dots,n\} \\ |A| = l}} \prod_{i\in A} a_i \le \binom{n}{l} \left(\frac{1}{n}\sum_{i=1}^n a_i\right)^l .$$

We will need the following consequence of this Lemma.

Corollary 3 *Let $1 \le k \le n$. Let a_i, $i = 1, \dots, n$ be nonnegative real numbers. Assume*

$$0 < a := \frac{e}{k} \sum_{i=1}^n a_i < 1.$$

Then

$$\sum_{l=k}^n \sum_{\substack{A \subset \{1,2,\dots,n\} \\ |A| = l}} \prod_{i\in A} a_i < \frac{1}{\sqrt{2\pi k}}\, \frac{a^k}{1-a}.$$

Proof. By Lemma 2 we have

$$\sum_{l=k}^{n} \sum_{\substack{A\subset\{1,2,\ldots,n\}\\ |A|=l}} \prod_{i\in A} a_i \le \sum_{l=k}^{n} \binom{n}{l} \left(\frac{ka}{en}\right)^l = \sum_{l=k}^{n} \frac{n! a^l k^l}{l!(n-l)! e^l n^l} \le \sum_{l=k}^{n} \frac{a^l k^l}{l! e^l}.$$

Applying Stirling's formula (5), we obtain

$$\sum_{l=k}^{n} \sum_{\substack{A\subset\{1,2,\ldots,n\}\\ |A|=l}} \prod_{i\in A} a_i \le \sum_{l=k}^{n} \frac{a^l k^l}{l^l \sqrt{2\pi l}} \le \frac{1}{\sqrt{2\pi k}} \sum_{l=k}^{n} a^l < \frac{1}{\sqrt{2\pi k}} \frac{a^k}{1-a}.$$

□

We will also need the following result.

Lemma 4 *Let $1 \le k \le n$. Let $(a_i)_{i=1}^n$, be a nonincreasing sequence of positive real numbers. Then there exists a partition $(A_l)_{l\le k}$ of $\{1,2,...,n\}$ such that*

$$\min_{1\le l\le k} \sum_{i\in A_l} a_i \ge a := \tfrac{1}{2} \min_{1\le j\le k} \frac{1}{k+1-j} \sum_{i=j}^{n} a_i.$$

Remark. In fact our proof gives that the A_l's can be taken as intervals, i.e. $A_l = \{i \mid n_{l-1} < i \le n_l\}$, $l \le k$, for some sequence $0 = n_0 < 1 \le n_1 < n_2 < \ldots < n_k = n$.

Proof. Denote $b := \sum_{i=1}^n a_i$.

Case 1. $a_1 \le b/k$. Let $n_0 = 0$ and, given $1 \le l \le k$, let n_l be the largest integer such that

$$\sum_{i=1}^{n_l} a_i \le \frac{lb}{k}.$$

Since $b/k \ge a_1 \ge a_2 \ge \ldots \ge a_n$, we have $0 = n_0 < 1 \le n_1 < n_2 < \ldots < n_k = n$. Define a partition $(A_l)_{l\le k}$ of $\{1,2,...,n\}$ by $A_l = \{i \mid n_{l-1} < i \le n_l\}$. Let t be the largest integer such that $a_t > \frac{b}{2k}$ (if there is no such a_t we put $t = 0$). Then

[i] for every l such that $n_l \le t$ we have $\sum_{i\in A_l} a_i \ge a_{n_l} > \frac{b}{2k}$;

[ii] for every $l < k$ such that $n_l > t$ we have $\sum_{i\in A_l} a_i \ge \frac{b}{2k}$ (otherwise, since $a_{n_l+1} \le \frac{b}{2k}$, we would have

$$\sum_{i=1}^{n_l+1} a_i \le \sum_{i=1}^{n_{l-1}} a_i + \sum_{i\in A_l} a_i + a_{n_l+1} < \frac{(l-1)b}{k} + \frac{b}{2k} + \frac{b}{2k} = \frac{lb}{k},$$

which contradicts the choice of n_l);

[iii] for $l = k$ we have $\sum_{i\in A_k} a_i = \sum_{i=1}^n a_i - \sum_{i=1}^{n_{k-1}} a_i \ge \frac{b}{k}$.

Since $\frac{b}{2k} \ge a$, it proves the result in Case 1.

Case 2. $a_1 > b/k$. Denote $b_j := \sum_{i=j}^n a_i$, $j \le n$. Clearly $a_k \le b_k$. Let $m \le k$ be the smallest integer such that

$$a_m \le \frac{b_m}{k+1-m}.$$

Since $a_1 > b/k = b_1/k$ we have $m > 1$. For $1 \le l < m$ choose $A_l = \{l\}$. Then

$$\sum_{i \in A_l} a_i = a_l > \frac{b_l}{k+1-l} > a.$$

Let $(A_l)_{l=m}^k$ be the partition of $\{m, m+1, \dots, n\}$ into $k+1-m$ sets constructed in the same way as in the Case 1. Then, by Case 1, for every $l \ge m$

$$\sum_{i \in A_l} a_i \ge \frac{b_m}{2(k+1-m)} \ge a.$$

It completes the proof. $\square$

Remark. One can show that the sequence

$$\bar{a}_j = \frac{1}{k+1-j} \sum_{i=j}^n a_i,$$

$j \le k$, considered in the last lemma (and which will appear again below) has the following properties: there exists an integer $1 \le r \le k$ such that

$$\bar{a}_1 > \bar{a}_2 > \ldots > \bar{a}_r \le \bar{a}_{r+1} \le \ldots \bar{a}_k. \tag{6}$$

and

$$\bar{a}_{j+1} \ge \bar{a}_j \quad \text{if and only if} \quad \sum_{i=j+1}^n a_i \ge (k-j) a_j. \tag{7}$$

To see (7), note that by the formula for $\bar{a}_j$'s we immediately obtain that $\bar{a}_{j+1} \ge \bar{a}_j$ if and only if

$$(k+1-j) \sum_{i=j+1}^n a_i \ge (k-j) \sum_{i=j}^n a_i = (k-j) \sum_{i=j+1}^n a_i + (k-j) a_j,$$

which implies (7). Now note that (7) implies that

$$\text{if} \quad \bar{a}_{j+1} \ge \bar{a}_j \quad \text{then} \quad \bar{a}_{j+2} \ge \bar{a}_{j+1}. \tag{8}$$

Indeed, if $\bar{a}_{j+1} \ge \bar{a}_j$ then, by (7), $(k-j)a_j \le \sum_{i=j+1}^n a_i$. Therefore, since $(a_i)_i$ is nonincreasing,

$$(k-j-1) a_{j+1} \le (k-j) a_j - a_{j+1} \le \sum_{i=j+1}^n a_i - a_{j+1} = \sum_{i=j+2}^n a_i,$$

which implies $\bar{a}_{j+2} \ge \bar{a}_{j+1}$ by (7). Finally, (6) is a simple consequence of (8).

4 Main results

In this section we state our main theorems, discussed in the introduction, and provide corresponding deviation inequalities which imply the main theorems.

Theorem 5 *Let $\alpha > 0$, $\beta > 0$. Let $p > 0$. Let $(x_i)_{i=1}^n$ be a sequence of real numbers and ξ_1, ..., ξ_n be random variables satisfying the (α, β)-condition. Then*

$$\tfrac{1}{1+p}\,\alpha^{-p}\left(\sum_{i=1}^n 1/|x_i|\right)^{-p} \leq \mathbb{E}\min_{1\leq i\leq n}|x_i\xi_i|^p.$$

Moreover, if the ξ_i's are independent then

$$\mathbb{E}\min_{1\leq i\leq n}|x_i\xi_i|^p \leq \beta^{-p}\,\Gamma(1+p)\left(\sum_{i=1}^n 1/|x_i|\right)^{-p}.$$

An immediate consequence of this theorem is the following Corollary.

Corollary 6 *Let $p > 0$. Let $(x_i)_{i=1}^n$ be a sequence of real numbers and $f_1, \dots, f_n$, $\xi_1, \dots, \xi_n$ be random variables satisfying the (α, β)-condition. Assume that the ξ_i's are independent. Then*

$$\mathbb{E}\min_{1\leq i\leq n}|x_i\xi_i|^p \leq \Gamma(2+p)\,\alpha^p\,\beta^{-p}\,\mathbb{E}\min_{1\leq i\leq n}|x_i f_i|^p.$$

In particular, if $f_1, \dots, f_n$, $\xi_1, \dots, \xi_n$ are $N(0,1)$ Gaussian random variables, then

$$\mathbb{E}\min_{1\leq i\leq n}|x_i\xi_i|^p \leq \Gamma(2+p)\,\mathbb{E}\min_{1\leq i\leq n}|x_i f_i|^p.$$

Theorem 7 *Let $\alpha > 0$, $\beta > 0$. Let $p > 0$ and $2 \leq k \leq n$. Let $0 < x_1 \leq x_2 \leq ... \leq x_n$ and $\xi_1, \dots, \xi_n$ be independent random variables satisfying the (α, β)-condition. Then*

$$c_{p\alpha}\max_{1\leq j\leq k}\frac{k+1-j}{\sum_{i=j}^n 1/x_i} \leq \left(\mathbb{E}\,k\text{-}\min_{1\leq i\leq n}|x_i\xi_i|^p\right)^{1/p} \leq \beta^{-1}\,C(p,k)\max_{1\leq j\leq k}\frac{k+1-j}{\sum_{i=j}^n 1/x_i},$$

where $c_{p\alpha} = \frac{1}{2e\alpha}\left(1 - \frac{1}{4\sqrt{\pi}}\right)^{1/p}$ and $C(p,k) = 4\sqrt{2}\,\max\{p, \ln(1+k)\}$.

Theorems 5 and 7 are consequences of the following Lemmas, which are of independent interest. In [GLSW3] we showed how these Lemmas imply the Theorems in the Gaussian case. Since the proof of the general case, which we consider here, is done in exactly the same way, we omit the proofs of the Theorems and concentrate on the proofs of the Lemmas.

Lemma 8 *Let $\alpha > 0$, $\beta > 0$. Let $0 < x_1 \leq x_2 \leq ... \leq x_n$ and $\xi_1, \dots, \xi_n$ be random variables satisfying the (α, β)-condition. Let $a = \sum_{i=1}^n 1/x_i$. Then for every $t > 0$*

$$\mathbb{P}\left\{\omega \,\middle|\, \min_{1\leq i\leq n}|x_i\xi_i(\omega)| \leq t\right\} \leq \alpha\,a\,t.$$

Moreover, if the ξ_i's are independent then for every $t > 0$

$$\mathbb{P}\left\{\omega \,\middle|\, \min_{1\leq i\leq n}|x_i\xi_i(\omega)| > t\right\} \leq e^{-\beta\,a\,t}.$$

Proof. Denote $A_k(t) = \{\omega \mid |x_k \xi_k(\omega)| > t\} = \{\omega \mid |\xi_k(\omega)| > t/x_k\}$ and

$$A(t) = \{\omega \mid \min_{k \le n} |x_k \xi_k(\omega)| > t\} = \bigcap_{k \le n} A_k(t).$$

By (1) we have $\mathbb{P}\left(A_k(t)^c\right) \le \alpha\, t/x_k$. Therefore

$$\mathbb{P}\left(A(t)\right) \ge 1 - \sum_{k=1}^{n} \mathbb{P}\left(A_k(t)^c\right) \ge 1 - \alpha\, t \sum_{k=1}^{n} 1/x_k,$$

which proves the first estimate.

Now assume that the ξ_k's are independent. By (2) we have $\mathbb{P}\left(A_k(t)\right) \le \exp\left(-\beta t/x_k\right)$. Therefore,

$$\mathbb{P}\left(A(t)\right) = \prod_{k=1}^{n} \mathbb{P}\left(A_k(t)\right) \le \exp\left(-\beta \sum_{k=1}^{n} t/x_k\right),$$

which proves the result. $\square$

Lemma 9 *Let $\alpha > 0$, $\beta > 0$. Let $1 \le k \le n$. Let $0 < x_1 \le x_2 \le \ldots \le x_n$ and $\xi_1, \ldots, \xi_n$ be independent random variables satisfying the (α, β)-condition. Let*

$$a = \frac{\alpha\, e}{k} \sum_{i=1}^{n} \frac{1}{x_i}.$$

Then for every $0 < t < 1/a$ one has

$$\mathbb{P}\left\{\omega \,\middle|\, k\text{-}\min_{1 \le i \le n} |x_i \xi_i(\omega)| \le t\right\} \le \frac{1}{\sqrt{2\pi k}}\, \frac{(at)^k}{1 - at}. \tag{9}$$

Proof. Denote $A(t) = \mathbb{P}\left\{\omega \mid k\text{-}\min_{1 \le i \le n} |x_i \xi_i(\omega)| \le t\right\}$. Clearly we have

$$\begin{aligned} A(t) &= \mathbb{P}\left\{\omega \,\middle|\, \exists i_1, \ldots, i_k \ge 1 : |\xi_{i_j}(\omega)| \le \frac{t}{x_{i_j}}\right\} \\ &= \mathbb{P} \bigcup_{\ell=k}^{n} \bigcup_{\substack{A \subset \{1,\ldots,n\} \\ |A| = \ell}} \left\{\omega | \forall i \in A : |\xi_i(\omega)| \le \frac{t}{x_i} \text{ and } \forall i \notin A : |\xi_i(\omega)| > \frac{t}{x_i}\right\} \\ &= \sum_{l=k}^{n} \sum_{\substack{A \subset \{1,\ldots,n\} \\ |A| = l}} \prod_{i \in A} \mathbb{P}\left\{\omega \mid |\xi_i(\omega)| \le \frac{t}{x_i}\right\} \prod_{i \notin A} \mathbb{P}\left\{\omega \mid |\xi_i(\omega)| > \frac{t}{x_i}\right\}. \end{aligned}$$

It follows

$$A(t) \le \sum_{l=k}^{n} \sum_{\substack{A \subset \{1,\ldots,n\} \\ |A| = l}} \prod_{i \in A} \mathbb{P}\left\{\omega \mid |\xi_i(\omega)| \le \frac{t}{x_i}\right\} \le \sum_{l=k}^{n} \sum_{\substack{A \subset \{1,\ldots,n\} \\ |A| = l}} \prod_{i \in A} \alpha\, \frac{t}{x_i}.$$

Corollary 3 implies the desired result. $\square$

5 Examples

In this section we provide some examples. They show that Theorem 7 is sharp. Namely, we show that both, the upper and lower estimate in Theorem 7 can be attained. We also provide an example where the actual value of $\mathbb{E}\ k$- min lies between the two estimates.

Example 10 *Let $1 \le k \le n$. Let g_i, $i \le n$, be independent $N(0,1)$ Gaussian random variables. Then*
(i) for $k \le n/2$

$$\sqrt{\frac{\pi}{2}}\ \frac{k}{n+1} \le \mathbb{E}\ k\text{-}\min_{1\le i\le n} |g_i| \le \sqrt{2\pi}\ \frac{k}{n+1}.$$

(ii) For $k \ge n/2$

$$c\sqrt{\ln\frac{2n}{n+1-k}} \le \mathbb{E}\ k\text{-}\min_{1\le i\le n} |g_i| \le C\sqrt{\ln\frac{2n}{n+1-k}},$$

where c and C are absolute positive constants.

Remark Note that in this case $x_1 = \cdots x_n = 1$, hence $\max_{1\le j\le k} \frac{k+1-j}{\sum_{i=j}^{n} 1/x_i} = k/n$. Thus (i) of this example shows that the lower estimate in Theorem 7 can be attained (up to an absolute constant).

Proof of Example 10. (i) follows immediately from the following claim:
Claim. For all $1 \le k \le n$ one has

$$\sqrt{\frac{\pi}{2}}\ \frac{k}{n+1} \le \mathbb{E}\ k\text{-}\min_{1\le i\le n} |g_i| \le \sqrt{\frac{\pi}{2}}\ \frac{k}{n+1-k}.$$

We now prove the claim. Denote $u = u(t) = \mathbb{P}\{|g_1| > t\} = \sqrt{\frac{2}{\pi}} \int_t^\infty e^{-s^2/2} ds$. Using the same ideas as in the proof of Lemma 9, we observe

$$\mathbb{E}\ k\text{-}\min_{1\le i\le n} |g_i| = \int_0^\infty \mathbb{P}\left\{k\text{-}\min_{1\le i\le n} |g_i| > t\right\} dt = \sum_{l=0}^{k-1} \sum_{\substack{A\subset\{1,\ldots,n\}\\ |A|=n-l}} \int_0^\infty \left(\prod_{i\in A} u(t) \prod_{i\notin A} (1-u(t))\right) dt$$

$$= \sum_{l=0}^{k-1} \binom{n}{l} \int_0^\infty (u(t))^{n-l} (1-u(t))^l\ dt = \sqrt{\frac{\pi}{2}} \sum_{l=0}^{k-1} \binom{n}{l} \int_0^\infty (u(t))^{n-l} (1-u(t))^l\ e^{t^2/2}\ (-du(t)).$$

To obtain the lower estimate note that $e^{t^2/2} \ge 1$ and therefore

$$\mathbb{E}\ k\text{-}\min_{1\le i\le n} |g_i| \ge \sqrt{\frac{\pi}{2}} \sum_{l=0}^{k-1} \binom{n}{l}\ B(n-l+1, l+1) = \sqrt{\frac{\pi}{2}}\ \frac{k}{n+1},$$

where $B(x,y) = \int_0^1 s^{x-1}(1-s)^{y-1} ds = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the Beta-function.

To obtain the upper estimate note that considering function $f(t) = e^{t^2/2}\int_x^\infty e^{-s^2/2}\, ds$ one can obtain that $u(t)e^{t^2/2} \le 1$, and therefore

$$\mathbb{E}\ k\text{-}\min_{1\le i\le n} |g_i| \le \sqrt{\frac{\pi}{2}} \sum_{l=0}^{k-1} \binom{n}{l}\ B(n-l, l+1) = \sqrt{\frac{\pi}{2}} \sum_{l=0}^{k-1} \frac{1}{n-l} \le \sqrt{\frac{\pi}{2}}\ \frac{k}{n+1-k}.$$

(ii) is well known. □

The next two examples can be obtained by direct calculation as well. One should split a sequence into two parts and use deviation inequalities. We omit the details.

Example 11 *Let $1 \le k \le n/2$. Let g_i, $i \le n$, be independent exponentially distributed random variables (i.e. with density $p(s) = e^{-s}$). Let $x_1 = \cdots = x_k = 1$, $x_{k+1} = \cdots = x_n = n^2$. Then*

$$c \ \ln k \le \mathbb{E}\ k\text{-}\min_{1 \le i \le n} |x_i g_i| \le C \ \ln k,$$

where c and C are absolute positive constants.

Remark Note that in this case $\max_{1 \le j \le k} \frac{k+1-j}{\sum_{i=j}^{n} 1/x_i} = \frac{k}{k+(n-k)/n^2} \approx 1$. Thus this example shows that the upper estimate in Theorem 7 can be attained (up to an absolute constant).

Example 12 *Let $1 \le k \le n$. Let g_i, $i \le n$, be independent $N(0,1)$ Gaussian random variables. Let $x_1 = \cdots = x_k = 1$, $x_{k+1} = \cdots = x_n = n^2$. Then*

$$c \ \sqrt{\ln k} \le \mathbb{E}\ k\text{-}\min_{1 \le i \le n} |x_i g_i| \le C \ \sqrt{\ln k},$$

where c and C are absolute positive constants.

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