Covering convex bodies by cylinders and lattice points by flats *

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Abstract

In connection with an unsolved problem of Bang (1951) we give a lower bound for the sum of the base volumes of cylinders covering a d-dimensional convex body in terms of the relevant basic measures of the given convex body. As an application we establish lower bounds on the number of k-dimensional flats (i.e. translates of k-dimensional linear subspaces) needed to cover all the integer points of a given convex body in d-dimensional Euclidean space for $1 \le k \le d-1$.

1 Introduction

In a remarkable paper [Ba] Bang has given an elegant proof of the plank conjecture of Tarski showing that if a convex body is covered by finitely many planks in d-dimensional Euclidean space, then the sum of the widths of the planks is at least as large as the minimal width of the body. A celebrated extension of Bang's theorem to d-dimensional normed spaces has been given by Ball in [B3]. In his paper Bang raises also the important related question whether the sum of the base areas of finitely many cylinders covering a 3-dimensional convex body is at least half of the minimum area of a 2-dimensional projection of the body. If true, then Bang's estimate is

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sharp due to a covering of a regular tetrahedron by two cylinders described in [Ba]. We investigate this challenging problem of Bang in d-dimensional Euclidean space. Our main result is Theorem 3.1 presented and proved in Section 3. As a special case, we get that the sum of the base areas of finitely many cylinders covering a 3-dimensional convex body is always at least one third of the minimum area 2-dimensional projection of the body.

In [BeH] Bezdek and Hausel has established a discrete version of Tarski's plank problem by asking for the minimum number of hyperplanes that can cover the integer points within a convex body in d-dimensional Euclidean space. Theorem 5.1 of Section 5 gives an improvement of their result, which under some conditions improves also the corresponding estimate of Talata [Ta]. A related but different problem of covering the lattice points within a convex body by linear subspaces was investigated in [BarHPT]. Last but not least, Theorem 3.1 combined with some additional ideas leads to a lower bound on the number of k-dimensional flats (i.e. translates of k-dimensional linear subspaces) needed to cover all the integer points of a given convex body in d-dimensional Euclidean space for $1 \leq k \leq d-1$. This is the topic of Section 4 and its main result, Theorem 4.1, actually improves the corresponding estimate of Talata [Ta].

2 Notation

In this paper we identify a d-dimensional affine space with $\mathbb{R}^d$. By $|\cdot|$ and $\langle \cdot, \cdot \rangle$ we denote the canonical Euclidean norm and the canonical inner product on $\mathbb{R}^d$. The canonical Euclidean ball and sphere in $\mathbb{R}^d$ are denoted by $\mathbf{B}_2^d$ and S^{d-1}. By a subspace we always mean a linear subspace.

By a convex body in $\mathbb{R}^d$ we always mean a compact convex set with non-empty interior. The interior of $\mathbf{K}$ is denoted by int$\mathbf{K}$. Let $\mathbf{K} \subset \mathbb{R}^d$ be a convex body with the origin 0 in its interior. We denote by $\mathbf{K}^\circ$ the polar of $\mathbf{K}$, i.e.

$$\mathbf{K}^\circ = \{x \mid \langle x, y \rangle \leq 1 \ \text{ for every } \ y \in \mathbf{K}\}\,.$$

The Minkowski functional of $\mathbf{K}$ (or the gauge of $\mathbf{K}$) is

$$||x||_{\mathbf{K}} = \inf\{\lambda > 0 \mid x \in \lambda\mathbf{K}\}.$$

If $\mathbf{K}$ is a centrally symmetric convex body with its center of symmetry at the origin, then $||x||_{\mathbf{K}}$ defines a norm on $\mathbb{R}^d$ with the unit ball $\mathbf{K}$.

The *Banach-Mazur distance* between two convex bodies $\mathbf{K}$ and $\mathbf{L}$ in $\mathbb{R}^d$ is defined by

$$d(\mathbf{K},\mathbf{L}) = \inf\{\lambda > 0 \ \mid \ a \in \mathbf{L},\ b \in \mathbf{K},\ \mathbf{L} - a \subset T(\mathbf{K} - b) \subset \lambda(\mathbf{L} - a)\},$$

where the infimum is taken over all linear operators $T : \mathbb{R}^d \to \mathbb{R}^d$. The Banach-Mazur distance between $\mathbf{K}$ and the Euclidean ball $\mathbf{B}_2^d$ we denote by $d_{\mathbf{K}}$. As it is well-known, John's Theorem ([J]) implies that for every $\mathbf{K}$, $d_{\mathbf{K}}$ is bounded by d, while for centrally-symmetric convex body $\mathbf{K}$, $d_{\mathbf{K}} \le \sqrt{d}$ (see e.g. [B1]).

Given a convex body $\mathbf{K}$ in $\mathbb{R}^d$ we denote its distance to symmetric bodies by

$$sd_{\mathbf{K}} := \inf\left\{\lambda > 0 \ \mid \ a \in \mathbb{R}^d,\ -(\mathbf{K} - a) \subset \lambda(\mathbf{K} - a)\right\}. \tag{1}$$

Clearly, $sd_{\mathbf{K}} \le d_{\mathbf{K}} \le d$. In fact, $sd_{\mathbf{K}}$ is one of the ways to measure the asymmetry of the convex body $\mathbf{K}$. We refer to [Gr] for the related discussion.

Let $\mathbf{K}$ be a convex body in $\mathbb{R}^d$. We denote its volume by $\mathrm{vol}(\mathbf{K})$. When we would like to emphasize that we take d-dimensional volume of a body in $\mathbb{R}^d$ we write $\mathrm{vol}_d(\mathbf{K})$.

Given a linear subspace (in short, a subspace) $E \subset \mathbb{R}^d$ we denote the orthogonal projection on E by P_E and the orthogonal complement of E by $E^\perp$. We will use the following theorem, proved by Rogers and Shephard ([RS], see also [C] and Lemma 8.8 in [Pi1]).

Theorem 2.1 *Let $1 \le k < d$. Let $\mathbf{K}$ be a convex body in $\mathbb{R}^d$ and E be a k-dimensional subspace of $\mathbb{R}^d$. Then*

$$\max_{x \in \mathbb{R}^d} \mathrm{vol}_{d-k}\left(\mathbf{K} \cap \left(x + E^\perp\right)\right) \mathrm{vol}_k(P_E \mathbf{K}) \le \binom{d}{k} \mathrm{vol}_d(\mathbf{K}).$$

Remark. Note that the reverse estimate

$$\max_{x \in \mathbb{R}^d} \mathrm{vol}_{d-k}\left(\mathbf{K} \cap \left(x + E^\perp\right)\right) \mathrm{vol}_k(P_E \mathbf{K}) \ge \mathrm{vol}_d(\mathbf{K})$$

is a simple application of the Fubini Theorem and is correct for any measurable set $\mathbf{K}$ in $\mathbb{R}^d$.

We will be using the following parameters of a convex body $\mathbf{K}$ with 0 in its interior

$$M(\mathbf{K}) := \int_{S^{d-1}} \|x\|_{\mathbf{K}} \, d\sigma(x),$$

where σ denotes the normalized Lebesgue measure on S^{d-1}, $M^*(\mathbf{K}) := M(\mathbf{K}^\circ)$, and

$$MM^*(\mathbf{K}) := \inf M(T(\mathbf{K}-a))M^*(T(\mathbf{K}-a)),$$

where the infimum is taken over all invertible linear maps $T : \mathbb{R}^d \to \mathbb{R}^d$ and all a in the interior of $\mathbf{K}$. Note that $M^*(\mathbf{K})$ is the half of mean width of $\mathbf{K}$. Below we need the following theorem.

Theorem 2.2 *There exist absolute positive constants C and α such that for every $d \geq 1$ and every convex body $\mathbf{K}$ in $\mathbb{R}^d$ one has*

$$MM^*(\mathbf{K}) \leq Cd^{1/3} \ln^\alpha(d+1).$$

Moreover, if $\mathbf{K}$ is centrally symmetric then

$$MM^*(\mathbf{K}) \leq C \ln(d+1).$$

The second estimate in this theorem is a well-known fact from Asymptotic Theory of finite dimensional normed spaces (see, e.g., [Pi1, To]). In fact, it is a combination of results by Lewis ([L]), by Figiel and Tomczak-Jaegermann ([FT]) with a deep theorem by Pisier on the so-called Rademacher projection ([Pi2]). The result in the general case is due to Rudelson ([Rud]). The both estimates of the theorem plays an essential role in the Asymptotic Theory.

The lattice width of a convex body $\mathbf{K}$ in $\mathbb{R}^d$ is defined as

$$\mathrm{w}(\mathbf{K}, \mathbb{Z}^d) = \min \left\{ \max_{x \in \mathbf{K}} \langle x, y \rangle - \min_{x \in \mathbf{K}} \langle x, y \rangle \ \mid \ y \in \mathbb{Z}^d, \ y \neq 0 \right\}.$$

Note that, if the origin is in the interior of $\mathbf{K}$, then

$$\mathrm{w}(\mathbf{K}, \mathbb{Z}^d) = \min \left\{ \|y\|_{\mathbf{K}^\circ} + \|-y\|_{\mathbf{K}^\circ} \ \mid \ y \in \mathbb{Z}^d, \ y \neq 0 \right\}.$$

The flatness parameter of $\mathbf{K}$ is defined as

$$\mathrm{Flt}(\mathbf{K}) = \sup \mathrm{w}(T\mathbf{K}, \mathbb{Z}^d),$$

where the supremum is taken over all invertible affine maps $\mathbb{R}^d \to \mathbb{R}^d$ satisfying $T\mathbf{K} \cap \mathbb{Z}^d = \emptyset$. The following theorem was proved in [Ban] for the centrally symmetric case and the case of an ellipsoid, and in [BanLPS] for the general case. It improves the previous bound by Kannan and Lovász ([KL]), who showed $\mathrm{Flt}(\mathbf{K}) \leq Cd^2$.

Theorem 2.3 *There exist absolute positive constants C and c such that for every $d \geq 1$ and every convex body $\mathbf{K}$ in $\mathbb{R}^d$ one has*

$$cd \leq \mathrm{Flt}(\mathbf{K}) \leq CdMM^*(\mathbf{K}).$$

Moreover, $\mathrm{Flt}(\mathbf{K}) \leq d$ *if* $\mathbf{K}$ *is an ellipsoid.*

3 Covering by cylinders

In this section we introduce a volumetric parameter related to covering by cylinders and provide corresponding estimates.

By a cylinder in $\mathbb{R}^d$ we always mean a 1-codimensional cylinder, that is, a set $C \subset \mathbb{R}^d$ that can be presented as $C = \ell + B$, where ℓ is a line containing 0 in $\mathbb{R}^d$ and B is a measurable set in $E := \ell^\perp$. Let $\mathbf{K} \subset \mathbb{R}^d$ be a convex body and $C \subset \mathbb{R}^d$ be a cylinder. The cross-sectional volume of C with respect to $\mathbf{K}$ we denote by

$$\mathrm{crv}_{\mathbf{K}}(C) := \frac{\mathrm{vol}_{d-1}(C \cap E)}{\mathrm{vol}_{d-1}(P_E \mathbf{K})} = \frac{\mathrm{vol}_{d-1}(P_E C)}{\mathrm{vol}_{d-1}(P_E \mathbf{K})} = \frac{\mathrm{vol}_{d-1}(B)}{\mathrm{vol}_{d-1}(P_E \mathbf{K})}.$$

It is easy to see that for every $(d-1)$-dimensional subspace $H \subset \mathbb{R}^d$ not containing ℓ one has

$$\mathrm{crv}_{\mathbf{K}}(C) = \frac{\mathrm{vol}_{d-1}(C \cap H)}{\mathrm{vol}_{d-1}(P\mathbf{K})},$$

where P is the projection on H with the kernel ℓ. We would also like to notice that for every invertible affine map $T : \mathbb{R}^d \to \mathbb{R}^d$ one has $\mathrm{crv}_{\mathbf{K}}(C) = \mathrm{crv}_{T\mathbf{K}}(TC)$.

Theorem 3.1 *Let $\mathbf{K}$ be a convex body in $\mathbb{R}^d$. Let $C_1, \ldots, C_N$ be cylinders in $\mathbb{R}^d$ such that*

$$\mathbf{K} \subset \bigcup_{i=1}^{N} C_i.$$

Then

$$\sum_{i=1}^{N} \mathrm{crv}_{\mathbf{K}}(C_i) \geq \frac{1}{d}.$$

Moreover, if $\mathbf{K}$ *is an ellipsoid then*

$$\sum_{i=1}^{N} \mathrm{crv}_{\mathbf{K}}(C_i) \geq 1.$$

Proof: In this proof we denote $v_n := \mathrm{vol}_n(\mathbf{B}_2^n)$. Every C_i can be presented as $C_i = \ell_i + B_i$, where ℓ_i is a line containing 0 in $\mathbb{R}^d$ and B_i is a body in $E_i := \ell_i^\perp$.

We first prove the theorem for ellipsoids. Since $\mathrm{crv}_{\mathbf{K}}(C) = \mathrm{crv}_{T\mathbf{K}}(TC)$ for every invertible affine map $T : \mathbb{R}^d \to \mathbb{R}^d$, we may assume that $\mathbf{K} = \mathbf{B}_2^d$. Then

$$\mathrm{crv}_{\mathbf{K}}(C_i) = \frac{\mathrm{vol}_{d-1}(B_i)}{v_{d-1}}.$$

Consider the following (density) function on $\mathbb{R}^d$

$$p(x) = 1/\sqrt{1 - |x|^2}$$

for $|x| < 1$ and $p(x) = 0$ otherwise. The corresponding measure on $\mathbb{R}^d$ we denote by μ, that is $d\mu(x) = p(x)dx$. Let ℓ be a line containing 0 in $\mathbb{R}^d$ and $E = \ell^\perp$. It follows from direct calculations that for every $z \in E$ with $|z| < 1$

$$\int_{\ell+z} p(x)\ dx = \pi.$$

Thus, we have

$$\mu(\mathbf{B}_2^d) = \int_{\mathbf{B}_2^d} p(x)\ dx = \int_{\mathbf{B}_2^d \cap E} \int_{\ell+z} p(x)\ dx\ dz = \pi\ v_{d-1}$$

and for every $i \leq N$

$$\mu(C_i) = \int_{C_i} p(x)\ dx = \int_{B_i} \int_{\ell_i+z} p(x)\ dx\ dz = \pi\ \mathrm{vol}_{d-1}\left(B_i\right).$$

Since $\mathbf{B}_2^d \subset \bigcup_{i=1}^N C_i$, we obtain

$$\pi\ v_{d-1} = \mu(\mathbf{B}_2^d) \leq \mu\left(\bigcup_{i=1}^{N} C_i\right) \leq \sum_{i=1}^{N} \mu\left(C_i\right) = \sum_{i=1}^{N} \pi\ \mathrm{vol}_{d-1}\left(B_i\right).$$

It implies

$$\sum_{i=1}^{N} \mathrm{crv}_{\mathbf{B}_2^d}(C_i) = \sum_{i=1}^{N} \frac{\mathrm{vol}_{d-1}(B_i)}{v_{d-1}} \geq 1. \tag{2}$$

Now, we show the general case. For $i \leq N$ denote $\bar{C}_i = C_i \cap \mathbf{K}$ and note that

$$\mathbf{K} \subset \bigcup_{i=1}^{N} \bar{C}_i \qquad \text{and} \qquad P_{E_i}\bar{C}_i = B_i \cap P_{E_i}\mathbf{K}.$$

Since $\bar{C}_i \subset \mathbf{K}$ we have also

$$\max_{x \in \mathbb{R}^d} \ \mathrm{vol}_1\left(\bar{C}_i \cap (x + \ell_i)\right) \leq \max_{x \in \mathbb{R}^d} \ \mathrm{vol}_1\left(\mathbf{K} \cap (x + \ell_i)\right).$$

Therefore, applying Theorem 2.1 (and Remark after it, saying that we don't need convexity of $\bar{C}_i$) we obtain for every $i \leq N$

$$\mathrm{crv}_{\mathbf{K}}(C_i) = \frac{\mathrm{vol}_{d-1}(B_i)}{\mathrm{vol}_{d-1}(P_{E_i}\mathbf{K})} \geq \frac{\mathrm{vol}_{d-1}(P_{E_i}\bar{C}_i)}{\mathrm{vol}_{d-1}(P_{E_i}\mathbf{K})}$$

$$\geq \frac{\mathrm{vol}_d(\bar{C}_i)}{\max_{x \in \mathbb{R}^d} \ \mathrm{vol}_1\left(\bar{C}_i \cap (x + \ell_i)\right)} \ \frac{\max_{x \in \mathbb{R}^d} \ \mathrm{vol}_1\left(\mathbf{K} \cap (x + \ell_i)\right)}{d\mathrm{vol}_d(\mathbf{K})} \geq \frac{\mathrm{vol}_d(\bar{C}_i)}{d\mathrm{vol}_d(\mathbf{K})}.$$

Using that $\bar{C}_i$'s covers $\mathbf{K}$, we observe

$$\sum_{i=1}^{N} \mathrm{crv}_{\mathbf{K}}(C_i) \geq \frac{1}{d},$$

which completes the proof. $\square$

Remark 1. If $\mathbf{K}$ is close to the Euclidean ball (and d is not very big), then the following estimate can be better than the general one

$$\sum_{i=1}^{N} \mathrm{crv}_{\mathbf{K}}(C_i) \geq \frac{1}{d_{\mathbf{K}}^{d-1}}.$$

It can be obtained as follows: Using that $\mathrm{crv}_{\mathbf{K}}(C) = \mathrm{crv}_{T\mathbf{K}}(TC)$ for an invertible affine transformation, we may assume that $\mathbf{B}_2^d$ is a distance ellipsoid for $\mathbf{K}$, namely assume that $\mathbf{B}_2^d \subset \mathbf{K} \subset d_{\mathbf{K}}\mathbf{B}_2^d$. Then

$$\sum_{i=1}^{N} \mathrm{crv}_{\mathbf{K}}(C_i) = \sum_{i=1}^{N} \frac{\mathrm{vol}_{d-1}(B_i)}{\mathrm{vol}_{d-1}(P_{E_i}\mathbf{K})} \geq \sum_{i=1}^{N} \frac{\mathrm{vol}_{d-1}(B_i)}{\mathrm{vol}_{d-1}(P_{E_i}d_{\mathbf{K}}\mathbf{B}_2^d)}$$

$$\geq d_{\mathbf{K}}^{-d+1} \sum_{i=1}^{N} \mathrm{crv}_{\mathbf{B}_2^d}(C_i) \geq d_{\mathbf{K}}^{-d+1}$$

(in the last inequality we used "moreover" part of Theorem 3.1). Recall that $d_{\mathbf{K}} \leq \sqrt{d}$ for any centrally symmetric convex body $\mathbf{K}$ in $\mathbb{R}^d$ and $d_{\mathbf{K}} \leq d$ in general. Thus, if $d = 3$ and $\mathbf{K}$ is a centrally-symmetric convex body, then this estimate is better than the general one given by Theorem 3.1.

Remark 2. Note that the proof of Theorem 3.1 can be extended to the case of cylinders of other dimensions. Indeed, given $k < d$ define a k-codimensional cylinder C as a set which can be presented in the form $C = H + B$, where H is a k-dimensional subspace of $\mathbb{R}^d$ and B is a measurable set in $E := H^{\perp}$. As before, given a convex body $\mathbf{K}$ and a k-codimensional cylinder $C = H + B$ denote

$$\mathrm{crv}_{\mathbf{K}}(C) := \frac{\mathrm{vol}_{d-k}(C \cap E)}{\mathrm{vol}_{d-k}(P_E \mathbf{K})} = \frac{\mathrm{vol}_{d-k}(P_E C)}{\mathrm{vol}_{d-k}(P_E \mathbf{K})} = \frac{\mathrm{vol}_{d-k}(B)}{\mathrm{vol}_{d-k}(P_E \mathbf{K})}.$$

Repeating the proof of Theorem 3.1 (the general case), we obtain that if a convex body $\mathbf{K}$ is covered by k-codimensional cylinders $C_1, \ldots, C_n$, then

$$\sum_{i=1}^{N} \mathrm{crv}_{\mathbf{K}}(C_i) \geq \frac{1}{\binom{d}{k}}.$$

As was noted by Bang ([Ba]), the case $k = d - 1$ here corresponds to the "plank problem", indeed, in this case we have the sum of relative widths of the body. As we mentioned in the introduction, Ball ([B3]) proved that such sum should exceed 1 in the case of centrally symmetric body $\mathbf{K}$, while the general case is still open. Our estimate implies the lower bound $1/d$. Of course, Ball's Theorem implies the estimate $1/sd_{\mathbf{K}}$.

4 Covering lattice points by lines and flats

Theorem 4.1 *Let $\mathbf{K}$ be a convex body in $\mathbb{R}^d$ containing the origin in its interior. Let $\ell_1, \ldots, \ell_N$ be lines in $\mathbb{R}^d$ such that*

$$\mathbf{K} \cap \mathbb{Z}^d \subset \bigcup_{i=1}^{N} \ell_i.$$

Then

$$N \geq \left(\frac{\mathrm{w}\left(\mathbf{K} \cap -\mathbf{K}, \mathbb{Z}^d\right)}{Cd\ MM^* \left(\mathbf{K} \cap -\mathbf{K}\right)} \right)^{d-1} \geq \left(\frac{\mathrm{w}\left(\mathbf{K} \cap -\mathbf{K}, \mathbb{Z}^d\right)}{C_0 d\ \ln(d+1)} \right)^{d-1},$$

where C and C_0 are absolute positive constants. If, in addition, $-\mathbf{K} \subset sd_{\mathbf{K}}\mathbf{K}$ (that is, if infimum in (1) attains at $a = 0$), then

$$N \geq \left(\frac{\mathrm{w}\left(\mathbf{K}, \mathbb{Z}^d\right)}{C\ sd_{\mathbf{K}}\ d\ MM^* \left(\mathbf{K}\right)} \right)^{d-1} \geq \left(\frac{\mathrm{w}\left(\mathbf{K}, \mathbb{Z}^d\right)}{C_0\ d^{7/3} \ln^{\alpha}(d+1)} \right)^{d-1},$$

where C, C_0, and α are absolute positive constants.

Moreover, if $\mathbf{K}$ is an ellipsoid centered at the origin, then

$$N \geq \left(\frac{\mathrm{w}\left(\mathbf{K}, \mathbb{Z}^d\right)}{2d} \right)^{d-1}.$$

Proof: Let $\lambda > 0$ be such that

$$\mathbf{K} \subset \bigcup_{i=1}^{N} (\ell_i + \lambda \mathbf{K}) \qquad \text{and} \qquad \mathbf{K} \not\subset \bigcup_{i=1}^{N} (\ell_i + \lambda\ \mathrm{int}\mathbf{K}).$$

Since $0 \in \mathbf{K}$, we have $0 \in l_i$ for some i, which clearly implies that $\lambda \leq 1$.

For $i \leq N$ let H_i denote the $(d-1)$-dimensional subspace orthogonal to ℓ_i and let P_i denote the orthogonal projection on H_i. We define

$$C_i := \ell_i + \lambda \mathbf{K} = \ell_i + \lambda P_i \mathbf{K}.$$

Then $\mathrm{crv}_{\mathbf{K}}(C_i) = \lambda^{d-1}$. Theorem 3.1 implies $N \geq c^d \lambda^{-d+1}$, where c is a positive absolute constant.

Now, $\mathbf{K} \not\subset \cup_{i=1}^{N} (\ell_i + \lambda\ \mathrm{int}\mathbf{K})$ if and only if there exists $x \in \mathbf{K}$ such that for every $i \leq N$ one has $x \notin \ell_i + \lambda\ \mathrm{int}\mathbf{K}$, i.e. $(x - \lambda\ \mathrm{int}\mathbf{K}) \cap \ell_i = \emptyset$. Let $y = (1 - \lambda/2)x$. By convexity of $\mathbf{K}$ we have

$$\left(y + \frac{\lambda}{2}\ (\mathbf{K} \cap -\ \mathrm{int}\mathbf{K}) \right) \subset \mathbf{K} \cap (x - \lambda\ \mathrm{int}\mathbf{K}).$$

Since $\mathbf{K} \cap \mathbb{Z}^d \subset \cup_{i=1}^{N} \ell_i$, we obtain

$$\left(y + \frac{\lambda}{2}\ (\mathbf{K} \cap -\ \mathrm{int}\mathbf{K}) \right) \cap \mathbb{Z}^d = \emptyset.$$

Using Theorem 2.3 (and, if needed, approximating λ by $\lambda - \varepsilon$ with small enough ε), we observe

$$\frac{\lambda}{2}\ \mathrm{w}\left(\mathbf{K}\cap -\mathbf{K}, \mathbb{Z}^d\right) = \mathrm{w}\left(y + \frac{\lambda}{2}\ \left(\mathbf{K}\cap -\mathbf{K}\right), \mathbb{Z}^d\right)$$

$$\leq \mathrm{Flt}(\mathbf{K}\cap -\mathbf{K}) \leq Cd\ MM^*\left(\mathbf{K}\cap -\mathbf{K}\right),$$

where C is an absolute constant. Thus,

$$N \geq c^d \lambda^{-d+1} \geq c^d \left(\frac{\mathrm{w}\left(\mathbf{K}\cap -\mathbf{K}, \mathbb{Z}^d\right)}{2Cd\ MM^*\left(\mathbf{K}\cap -\mathbf{K}\right)}\right)^{d-1}.$$

This shows the left-hand side of the first estimate. The right-hand side follows by Theorem 2.2. Note that in the case of ellipsoid we have $C = c = 1$, $MM^*\left(\mathbf{K}\cap -\mathbf{K}\right) = 1$, which implies the "moreover" part of the theorem.

The second estimate follows the same lines. For the sake of completeness we sketch it. Let $0 < \lambda \leq sd_{\mathbf{K}}$ be such that

$$\mathbf{K} \subset \bigcup_{i=1}^{N} (\ell_i - 2\lambda \mathbf{K}) \qquad \text{and} \qquad \mathbf{K} \not\subset \bigcup_{i=1}^{N} (\ell_i - \lambda\ \mathrm{int}\mathbf{K}).$$

Repeating arguments of the first part we obtain that $N \geq c^d \lambda^{-d+1}$ and $(x + \lambda\ \mathrm{int}\mathbf{K}) \cap \ell_i = \emptyset$ for every $i \leq N$. Convexity of $\mathbf{K}$ and the inclusion $-\mathbf{K} \subset sd_{\mathbf{K}}\mathbf{K}$ yields for $y = (1 - \lambda/(sd_{\mathbf{K}} + 1))x$

$$\left(y + \frac{\lambda}{sd_{\mathbf{K}} + 1}\ \ \mathrm{int}\mathbf{K}\right) \subset \mathbf{K} \cap (x + \lambda\ \mathrm{int}\mathbf{K}).$$

It implies

$$\left(y + \frac{\lambda}{sd_{\mathbf{K}} + 1}\ \ \mathrm{int}\mathbf{K}\right) \cap \mathbb{Z}^d = \emptyset$$

and, by Theorem 2.3,

$$\frac{\lambda}{sd_{\mathbf{K}} + 1}\ \mathrm{w}\left(\mathbf{K}, \mathbb{Z}^d\right) \leq C_1 d\ MM^*\left(\mathbf{K}\right).$$

Therefore,

$$N \geq c^d \lambda^{-d+1} \geq c^d \left(\frac{\mathrm{w}\left(\mathbf{K}, \mathbb{Z}^d\right)}{C_1\left(sd_{\mathbf{K}} + 1\right) d\ MM^*\left(\mathbf{K}\right)}\right)^{d-1},$$

which proves the left-hand estimate (with $C = 2C_1$). Since $sd_{\mathbf{K}} \leq d$, Theorem 2.2 implies the right-hand side inequality. $\square$

Remark. It is not difficult to see that the proof above can be extended almost without changes to the case of k-dimensional flats instead of lines (one needs to use Remark 2 following Theorem 3.1). In particular, for a centrally symmetric body $\mathbf{K} = -\mathbf{K}$, whose integer points are covered by the k-dimensional flats $H_1, \ldots, H_N$ we have

$$N \geq \left(\frac{\mathrm{w}\left(\mathbf{K}, \mathbb{Z}^d\right)(d-k)}{C\ d^2\ \ln(d+1)} \right)^{d-k}.$$

We omit the details and precise estimates in the non-symmetric case.

5 Covering lattice points by hyperplanes

The following theorem improves the estimate of the remark after Theorem 4.1 in the case $k = d-1$, extending a Bezdek-Hausel result from [BeH].

Theorem 5.1 *Let $\mathbf{K}$ be a centrally symmetric (with respect to the origin) convex body in $\mathbb{R}^d$. Let $H_1, \ldots, H_N$ be hyperplanes in $\mathbb{R}^d$ such that*

$$\mathbf{K} \cap \mathbb{Z}^d \subset \bigcup_{i=1}^{N} H_i.$$

Then

$$N \geq c\ \frac{\mathrm{w}(\mathbf{K}, \mathbb{Z}^d)}{d\ MM^*(\mathbf{K})} \geq c_0\ \frac{\mathrm{w}(\mathbf{K}, \mathbb{Z}^d)}{d\ \ln(d+1)},$$

where c, c_0 are absolute positive constants.

Proof: The proof is based on the Ball's solution of the plank problem. Namely, we use that given a centrally symmetric body $\mathbf{K} \subset \mathbb{R}^d$ and N hyperplanes $H_1, \ldots, H_N$ in $\mathbb{R}^d$ there exists $x \in R^d$ such that

$$\mathbf{L} := x + \frac{1}{N+1}\ \mathbf{K} \subset \mathbf{K}$$

and the interior of $\mathbf{L}$ is not met by any H_i (see Corollary or abstract in [B3]).

Since all integer points of $\mathbf{K}$ are covered by H_i's, we observe that

$$\mathrm{int}\mathbf{L} \cap \mathbb{Z}^d = \emptyset.$$

Applying Theorem 2.3, we obtain

$$\frac{1}{N+1}\,\mathrm{w}\left(\mathbf{K},\mathbb{Z}^d\right) = \mathrm{w}\left(\mathbf{L},\mathbb{Z}^d\right) \leq \mathrm{Flt}(\mathbf{K}) \leq Cd\; MM^*\left(\mathbf{K}\right),$$

where C is an absolute constant. Together with Theorem 2.2 it implies the desired result. □

References

[B1] K. Ball, *Flavors of geometry* in *An elementary introduction to modern convex geometry*, Levy, Silvio (ed.), Cambridge: Cambridge University Press. Math. Sci. Res. Inst. Publ. 31, 1–58 (1997).

[B2] K. Ball, *Volume ratios and a reverse isoperimetric inequality*, J. Lond. Math. Soc., II. Ser. 44 (1991), 351–359.

[B3] K. Ball, *The plank problem for symmetric bodies*, Invent. Math. 104 (1991), 535–543.

[Ba] T. Bang, *A solution of the "plank problem"*, Proc. Amer. Math. Soc. 2 (1951), 990–993.

[Ban] Banaszczyk, W. *Inequalities for convex bodies and polar reciprocal lattices in $\mathbb{R}^n$ II. Application of K-convexity*, Discrete Comput. Geom. 16 (1996), no. 3, 305–311.

[BanLPS] W. Banaszczyk, A. E. Litvak, A. Pajor, S. J. Szarek, *The flatness theorem for nonsymmetric convex bodies via the local theory of Banach spaces*, Math. Oper. Res. 24 (1999), no. 3, 728–750.

[BarHPT] I. Bárány, G. Harcos, J. Pach, G. Tardos, *Covering lattice points by subspaces*, Period. Math. Hung. 43 (2001), 93–103.

[BeH] K. Bezdek, T. Hausel, *On the number of lattice hyperplanes which are needed to cover the lattice points of a convex body*, Böröczky, K. (ed.)

et al., Intuitive geometry. Proceedings of the 3rd international conference held in Szeged, Hungary, 1991. Amsterdam: North-Holland. Colloq. Math. Soc. János Bolyai. 63 (1994), 27–31.

[C] G. D. Chakerian, *Inequalities for the difference body of a convex body*, Proc. Amer. Math. Soc. 18 (1967), 879–884.

[FT] T. Figiel, N. Tomczak-Jaegermann, *Projections onto Hilbertian subspaces of Banach spaces*, Isr. J. Math. 33 (1979), 155–171.

[Gr] B. Grünbaum, *Measures of symmetry for convex sets.* 1963 Proc. Sympos. Pure Math., Vol. VII pp. 233–270 Amer. Math. Soc., Providence, R.I.

[J] F. John, *Extremum problems with inequalities as subsidiary conditions*, Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, 187–204. Interscience Publishers, Inc., New York, N. Y., 1948.

[KL] R. Kannan, L. Lovász, *Covering minima and lattice-point-free convex bodies*, Ann. of Math. 128 (1988), 577–602.

[L] D. R. Lewis, *Ellipsoids defined by Banach ideal norms*, Mathematika 26 (1979), 18–29.

[Pi1] G. Pisier, *Holomorphic semi-groups and the geometry of Banach spaces*, Ann. Math. 115 (1982), 375–392.

[Pi2] G. Pisier, *The Volume of Convex Bodies and Banach Space Geometry*, Cambridge University Press 1989.

[RS] C. A. Rogers, G. C. Shephard, *Convex bodies associated with a given convex body*, J. London Math. Soc. 33 (1958), 270–281.

[Rud] M. Rudelson, *Distances between non-symmetric convex bodies and the MM^*-estimate*, Positivity 4 (2000), 161–178.

[Ta] I. Talata, *Covering the lattice points of a convex body with affine subspaces*, Bolyai Soc. Math. Stud. 6 (1997), 429–440.

[To] N. Tomczak-Jaegermann, *Banach-Mazur distances and finite-dimensional operator ideals*, Pitman Monographs and Surveys in Pure and Applied Mathematics, 38. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1989.

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