Analyse fonctionnelle/Functional Analysis

Smallest singular value of random matrices with independent columns

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Abstract. We study the smallest singular value of a square random matrix with i.i.d. columns drawn from an isotropic symmetric log-concave distribution. We prove a deviation inequality in terms of the isotropic constant of the distribution.

Sur la plus petite valeur singulière de matrices aléatoires avec des colonnes indépendantes

Résumé. On étudie la plus petite valeur singulière d'une matrice carrée aléatoire dont les colonnes sont des vecteurs aléatoires i.i.d. suivant une loi à densité log-concave isotrope. On démontre une inégalité de déviation en fonction de la constante d'isotropie.

The behaviour of the smallest singular value of random matrices with i.i.d. random entries attracted a lot of attention over the years. Major results were recently obtained in [5, 8, 9, 10]. In asymptotic geometry one is interested in sampling vectors uniformly distributed in a convex body. In particular the entries are not necessarily independent. In this note, we study the more general case when the columns are i.i.d. random vectors with a symmetric isotropic log-concave distribution. We prove a deviation inequality for the smallest singular value in terms of a parameter L_μ which, in the case of sampling from a convex body, corresponds to the isotropic constant of the body.

Recall that a non-negative function f on $\mathbb{R}^n$ is called log-concave if for all $x, y \in \mathbb{R}^n$ and all $\theta \in (0,1)$, $f((1-\theta)x + \theta y) \geq f(x)^{1-\theta} f(y)^\theta$. In this paper a symmetric probability measure μ on $\mathbb{R}^n$ is said to be log-concave if its density f is symmetric log-concave and it is called isotropic if its covariance matrix is the identity. We will also set $L_\mu = f(0)^{1/n}$. Let us observe that if μ is an isotropic probability measure uniformly distributed on a symmetric convex body K then L_μ is the

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so-called isotropic constant of K. If X is a random vector, distributed according to μ, we will also write $L_X = L_\mu$.

We shall use the notation $|\cdot|$ to denote the Euclidean norm of a vector or the volume or the cardinality of a set.

Theorem 1 *Let $n \geq 1$ and let Γ is an $n \times n$ matrix with independent columns drawn from an isotropic symmetric log-concave probability μ. For every $\varepsilon \in (0,1)$ and all $\delta \in (0,1)$ and all $M \geq 1$ we have*

$$\mathbb{P}\Big(\inf_{x \in S^{n-1}} |\Gamma x| \leq \varepsilon \Big(\frac{c_1}{ML_\mu}\Big)^{\frac{1}{1-\delta}} n^{-1/2}\Big) \leq \frac{C\varepsilon}{\delta} + e^{-c_2 n} + \mathbb{P}(\|\Gamma\| > M\sqrt{n}), \tag{1}$$

where $c_1, c_2 > 0$ and C are absolute constants. Moreover, if $\delta \leq 1 - 1/(2n)$, then

$$\mathbb{P}\Big(\inf_{x \in S^{n-1}} |\Gamma x| \leq \varepsilon \Big(\frac{c_1}{ML_\mu}\Big)^{\frac{1}{1-\delta}} n^{-1/2}\Big) \leq \frac{C\varepsilon^{1/2}}{\delta} + \mathbb{P}(\|\Gamma\| > M\sqrt{n}). \tag{2}$$

Estimates for $\mathbb{P}(\|\Gamma\| > M\sqrt{n})$, when M is a power of $\log n$, can be deduced from [6] and [3].

An important case when we have more information (that follows from a result of Aubrun [1]) is that of 1-unconditional measures. Recall that a probability measure with density f is 1-unconditional if for any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and any $(\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, 1\}^n$, $f(x_1, \ldots, x_n) = f(\varepsilon_1 x_1, \ldots, \varepsilon_n x_n)$.

Corollary 2 *If a probability μ is 1-unconditional, then Γ satisfies*

$$\mathbb{P}(\inf_{x \in S^{n-1}} |\Gamma x| \leq \varepsilon n^{-1/2}) \leq C\varepsilon + 2e^{-cn^{1/5}},$$

where C and $c > 0$ are absolute constants. Moreover, for all $\varepsilon \in (0,1)$ we have

$$\mathbb{P}(\inf_{x \in S^{n-1}} |\Gamma x| \leq \varepsilon n^{-1/2}) \leq C\varepsilon^{cn^{1/5}/(2(cn^{1/5}+1))}$$

The proof of the theorem requires the study of the isotropic constant of a sum of i.i.d. random vectors in $\mathbb{R}^n$. Let $X_1, \ldots, X_n$ be independent isotropic log-concave symmetric random vectors in $\mathbb{R}^n$. Let $x \in S^{n-1}$, and set

$$Z = x_1 X_1 + \ldots + x_n X_n.$$

Then it is well-known that Z is also an isotropic log-concave symmetric random vector in $\mathbb{R}^n$. If $X_1, \ldots, X_n$ are 1-unconditional, then so is Z. The following theorem is of independent interest.

Theorem 3 *Let $X_1, \ldots, X_n$ are i.i.d. random vectors in $\mathbb{R}^n$, distributed according to a symmetric isotropic log-concave probability μ, let $x \in S^{n-1}$ and $Z = x_1 X_1 + \ldots + x_n X_n$. Then $L_Z \leq CL_\mu$, where C is a universal constant.*

The proof is based on the following version of a result by Gluskin and Milman [2]. Recall that K is called a star body whenever $tK \subset K$ for all $0 \leq t \leq 1$, and in such a case $\|\cdot\|_K$ denotes its Minkowski functional.

Lemma 4 *Let $f_1, \ldots, f_m$ be densities of probability measures on $\mathbb{R}^n$ and let $K \subset \mathbb{R}^n$ be a star body containing the origin in its interior. Then for all $\lambda_1, \ldots, \lambda_m$ we have*

$$\left(\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \Big\|\sum_{i=1}^m \lambda_i x_i\Big\|_K^2 \prod_{i=1}^m f(x_i) dx_i\right)^{1/2} \geq c|K|^{-1/n} \left(\sum_{i=1}^m \lambda_i^2 r_i^2\right)^{1/2}, \tag{3}$$

where $r_i^2 = \int_0^\infty |\{x \colon f_i(x) \geq t\}|^{1+2/n} dt \geq \|f_i\|_\infty^{-2/n}$ and $c > 0$ is an absolute constant.

Proof of Theorem 3. Let f be the density of μ and let g be the density of Z. By Lemma 2 in [4] there exists a star-shaped body $K \subset \mathbb{R}^n$, with 0 in its interior such that

$$g(0)^{1/n}|K|^{1/n}\left(\int_{\mathbb{R}^n}\|x\|_K^2 g(x)dx\right)^{1/2} \leq C,$$

for a certain universal constant C. On the other hand, by Lemma 4 we have

$$\begin{aligned}\left(\int_{\mathbb{R}^n}\|x\|_K^2 g(x)dx\right)^{1/2} &= \left(\mathbb{E}\|Z\|_K^2\right)^{1/2} = \left(\mathbb{E}\|x_1X_1+\ldots x_nX_n\|_K^2\right)^{1/2} \\ &\geq \frac{c}{|K|^{1/n}f(0)^{1/n}}\Big(\sum_{i=1}^n x_i^2\Big)^{1/2} = \frac{c}{|K|^{1/n}f(0)^{1/n}}.\end{aligned}$$

Putting these two inequalities together concludes the proof. $\square$

We pass now directly to the proof of Theorem 1 and we assume that Γ and μ satisfy the assumptions described there. Similarly as in [5, 8, 9], the argument relies on splitting the sphere S^{n-1} into several regions. We use the following notation from [9].

$$\begin{aligned}Sparse &= Sparse(\delta) = \{x\in\mathbb{R}^n:\ |\mathrm{supp}(x)|\leq\delta n\}\\ Comp &= Comp(\delta,\rho) = \{x\in S^{n-1}:\ \mathrm{dist}(x,Sparse(\delta))\leq\rho\}\\ Incomp &= Incomp(\delta,\rho) = S^{n-1}\setminus Comp(\delta,\rho)\end{aligned}$$

Proposition 5 *For all $\rho,\delta,\varepsilon\in(0,1)$ we have*

$$\mathbb{P}(\inf_{x\in Incomp(\delta,\rho)}|\Gamma x|\leq\rho\varepsilon n^{-1/2})\leq\frac{C}{\delta}\varepsilon$$

where C is an absolute constant.

The proof of this proposition uses Lemma 3.5 of [9] which reduces the required estimate to an estimate of probability of the form $\mathbb{P}_{X_k}(|\langle X_k^*,X_k\rangle|<\varepsilon)$, for a fixed $1\leq k\leq n$, where X_k^* is a random vector of norm 1 independent on X_k. For each fixed value of X_k^*, $\langle X_K^*,X_k\rangle$ is a one-dimensional isotropic log-concave and symmetric random variable and therefore the latter probability can be bounded above by $C\varepsilon$ where C is a universal constant. The proof is then finished by Lemma 3.5 of [9].

Proposition 6 *Let Γ be an $n\times n$ random matrix with independent columns $X_1,\ldots,X_n$, distributed according to a symmetric isotropic log-concave probability μ. Then, for any $M>1$ and $\delta,\rho\in(0,1)$, we have*

$$\mathbb{P}(\inf_{x\in Comp(\delta,\rho/(2M))}|\Gamma x|\leq\rho\sqrt{n}\ \&\ \|\Gamma\|\leq M\sqrt{n})\leq C^nL_\mu^nM^{\delta n}\rho^{(1-\delta)n},$$

where C is an absolute constant. In particular, there exist constants $c_1,c_2>0$ such that for every $M>1$ and $\delta,\rho\in(0,1)$, satisfying

$$\rho\leq\left(\frac{c_1}{M^\delta L_\mu}\right)^{\frac{1}{1-\delta}}$$

we have

$$\mathbb{P}(\inf_{x\in Comp(\delta,\rho/(2M))}|\Gamma x|\leq\rho\sqrt{n}\ \&\ \|\Gamma\|\leq M\sqrt{n})\leq e^{-c_2n}.$$

It is easy to see that for every fixed $x \in S^{n-1}$, letting $Z = \Gamma x$, we get

$$\mathbb{P}(|Z| \le \rho\sqrt{n}) \le C^n L_Z^n \rho^n,$$

where C is an absolute constant. Then the proof of Proposition 6 uses Theorem 3 and an ε-net argument. More sophisticated estimates for a small ball probability for random vectors distributed according to a symmetric isotropic log-concave measure were recently proved by Paouris [7].

Proof of Theorem 1. For a fixed $\delta \in (0,1)$ and $M \ge 1$, we apply Proposition 6 with

$$\rho = \Big(\frac{c_1}{M^\delta L_\mu}\Big)^{\frac{1}{1-\delta}}$$

and get

$$\mathbb{P}\Big(\inf_{x \in Comp(\delta, \rho/(2M))} |\Gamma x| \le \Big(\frac{c_1}{M^\delta L_\mu}\Big)^{\frac{1}{1-\delta}} \sqrt{n}\Big) \le e^{-c_2 n} + \mathbb{P}(\|\Gamma\| > M\sqrt{n}).$$

Since

$$\varepsilon\Big(\frac{c_1}{M L_\mu}\Big)^{\frac{1}{1-\delta}} n^{-1/2} = \varepsilon M^{-1}\Big(\frac{c_1}{M^\delta L_\mu}\Big)^{\frac{1}{1-\delta}} n^{-1/2} \le \Big(\frac{c_1}{M^\delta L_\mu}\Big)^{\frac{1}{1-\delta}} \sqrt{n},$$

we also have

$$\mathbb{P}\Big(\inf_{x \in Comp(\delta, \rho/(2M))} |\Gamma x| \le \varepsilon\Big(\frac{c_1}{M L_\mu}\Big)^{\frac{1}{1-\delta}} n^{-1/2}\Big) \le e^{-c_2 n} + \mathbb{P}(\|\Gamma\| > M\sqrt{n}).$$

Now, Proposition 5, applied with $\rho/2M$ instead of ρ and 2ε instead of ε, gives

$$\mathbb{P}\Big(\inf_{x \in Incomp(\delta, \rho/(2M))} |\Gamma x| \le \varepsilon\Big(\frac{c_1}{M L_\mu}\Big)^{\frac{1}{1-\delta}} n^{-1/2}\Big) \le \frac{C\varepsilon}{\delta}.$$

The last two inequalities combined with the fact that $S^{n-1} = Incomp(\delta, \rho/(2M)) \cup Comp(\delta, \rho/(2M))$ and union bound allow us to conclude (1).

The proof of the "moreover part" is similar. We omit further details. $\square$

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