The diameter of Minkowski compactum, random projections of convex bodies

Nicole Tomczak-Jaegermann

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based on [MT2] and [MT1] by Piotr Mankiewicz and NTJ

1 The asymptotic growth of the diameter of Minkowski compactum

For two n-dimensional normed spaces $X = (\mathbb{R}^n, B')$ and $Y = (\mathbb{R}^n, B'')$ with unit balls B', B'', respectively the Banach–Mazur distance is defined by

$$\mathrm{d}(X,Y) = \mathrm{d}(B', B'') = \inf\big\{\|T : B' \to B''\| \; \|T^{-1} : B'' \to B'\|\big\},$$

with the infimum taken over all invertible operators $T \in L(\mathbb{R}^n)$. (Here we adopt a convenient but not very usual notation

$$\|T : B' \to B''\| = \sup\{\|Tx\|_{B''} \mid \|x\|_{B'} \le 1\}$$

is the operator norm of T from X to Y.)

The Minkowski compactum $\mathcal{M}_n$ is the set of all n-dimensional Banach spaces equipped with the Banach–Mazur distance. (Strictly speaking, $\mathcal{M}_n$ is the set of *equivalence classes* of n-dimensional Banach spaces, with isometric spaces being identified.) From John's theorem, $\mathrm{d}(X, \ell_2^n) \le \sqrt{n}$, and thus the diameter $\operatorname{diam} \mathcal{M}_n = \sup_{X,Y} \mathrm{d}(X,Y) \le n$. However, the natural question about the lower bound for diam $\mathcal{M}_n$ was widely open until 1980. The breakthrough was made by Gluskin who introduced random finite-dimensional spaces in order to show in [G1] that the diameter of the Minkowski compactum $\mathcal{M}_n$ is asymptotically of order n. We present the complete proof of this result in the Gaussian setting.

By $g \in \mathbb{R}^n$ we denote the Gaussian random vector with $N(0, \frac{1}{n} I_n)$ distribution. The density of g is equal to

$$(n/2\pi)^{n/2} \exp(-n|x|^2/2).$$

Since $\mathbb{E}|g|^2 = 1$ we call g *a normalized* Gaussian vector (by $|\cdot|$ we denote the Euclidean norm on $\mathbb{R}^n$).

Basic properties of Gaussian vectors fundamental in the proof (see e.g., [DS]).

Fact 1 *Let $g \in \mathbb{R}^n$ be a normalized Gaussian random vector. Then for every Borel measurable set $B \subset H$,*

$$\mathbb{P}\{\omega \in \Omega \mid g(\omega) \in B\} \le e^{n/2} \operatorname{vol} B / \operatorname{vol} B_2^n. \tag{1}$$

Furthermore, for any $a, b > 0$ we have

$$\mathbb{P}\{\omega \in \Omega \mid |g(\omega)| \le a\} \ge 1 - (\sqrt{2} e^{-a^2/4})^n \tag{2}$$

and

$$\mathbb{P}\{\omega \in \Omega \mid |g(\omega)| \ge 1/b\} \ge 1 - (\sqrt{e}/b)^n. \tag{3}$$

Outline of the Proof. The proof of (1) is very simple. Observe that

$$\begin{aligned} \mathbb{P}\{\omega \in \Omega \mid g(\omega) \in B\} &= (n/2\pi)^{n/2} \int_B \exp(-n|x|^2/2) dx \\ &\le (n/2\pi)^{n/2} \int_B dx \le C^n \operatorname{vol} B / \operatorname{vol} B_2^n. \end{aligned} \tag{4}$$

Using $\operatorname{vol} B_2^n \sim n^{-n/2}$ we get some numerical constant C; to get factor $e^{n/2}$ use the formula for the volume of B_2^n and Stirling's formula,

$$\operatorname{vol} B_2^n = \pi^{n/2} / \Gamma(1 + n/2).$$

Estimate (3) immediately follows from (4). The following short proof of (2) is taken from [MT2]. Write a normalized Gaussian vector g in $\mathbb{R}^n$ in the form $g = n^{-1/2} \sum_{i=1}^n h_i e_i$, where h_i are standard $N(0,1)$ distributed independent Gaussian variables. Fix an arbitrary $\lambda \in (0, 1/2)$. Setting $y = \sqrt{1-2\lambda}t$ we have

$$(2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{(\lambda - 1/2)t^2} dt = (1-2\lambda)^{-1/2} (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-y^2/2} dy = (1-2\lambda)^{-1/2}.$$

Hence for every $a > 0$ we have

$$\begin{aligned} & \mathbb{P}\{\ \omega \in \Omega \mid \|g(\omega)\|_2 \geq a\} = \mathbb{P}\{\omega \in \Omega \mid \sum_{i=1}^n h_i^2(\omega) \geq a^2 n\} \\ \leq\ & e^{-a^2 \lambda n} \int_\Omega e^{\lambda \sum_{i=1}^n h_i^2(\omega)} d\mathbb{P}(\omega) = e^{-a^2 \lambda n} \prod_{i=1}^n (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{(\lambda - 1/2)x_i^2} dx_i \\ \leq\ & \left(e^{-a^2\lambda} (1-2\lambda)^{-1/2}\right)^n. \end{aligned} \tag{5}$$

Letting $\lambda = 1/4$, we get (2). $\square$

Let $N \geq 1$. Let $g_1, \ldots, g_N$ be independent normalized Gaussian vectors distributed as g; for convenience denote the underlying probability spaces by $\Omega_1, \ldots, \Omega_N$. We also let $\Omega = \Omega_1 \otimes \ldots \otimes \Omega_N$.

Gluskin's random polytope K_N in $\mathbb{R}^n$ is defined by

$$K_N(\omega) = \operatorname{conv}\{\pm e_1, \ldots, \pm e_n, \pm g_1(\omega_1), \ldots, \pm g_N(\omega_N)\}, \tag{6}$$

for $\omega = (\omega_1, \ldots, \omega_N) \in \Omega$.

Now let $g_1', \ldots, g_N'$, and $g_1'', \ldots, g_N''$ be independent copies of g, on the corresponding probability spaces Ω_i''s and Ω_i'''s; and consider appropriate Ω' and Ω'' similar as above.

To avoid sometimes cumbersome dependence of the estimates on the parameters, we shall assume from now on that $N = n$. Still, in many places we will write N to emphasize the number of random vertices rather than the dimension of the space.

(We should also mention that in some other problems some larger values of N might be needed, like $N \sim n^2$ or some other powers of n.)

Theorem 2 ([G1]) *There exist constants $c > 0$ and $C \geq 1$ such that for $n \geq 1$ and $N = n$ the following is true: The set of pairs $(\omega'.\omega'') \in \Omega' \otimes \Omega''$ such that*

$$\mathrm{d}\big(K'_N(\omega'), K''_N(\omega'')\big) < cn$$

has probability less than $CN \exp(-cn)$. Therefore, with probability close to 1, $\mathrm{d}\big(K'_N(\omega'), K''_N(\omega'')\big) \geq cn$.

First we pass to subsets $\Omega'_0 \subset \Omega'$ and $\Omega''_0 \subset \Omega''$ such that all Gaussian vectors g'_i and g''_i (for $1 \leq i \leq N$) have the Euclidean norms less than or equal to 2. (By (2), probability of the subset of Ω' on which $|g'_1| \geq 2$ is less than $\exp(-cn)$, therefore removing from Ω' all such subsets for $i = 1, \ldots, N$ results in the estimate $\mathbb{P}'(\Omega'_0) \geq 1 - Ne^{-cn}$. Also, $\mathbb{P}'(\Omega''_0) \geq 1 - Ne^{-cn}$.) The rest of the proof will be done on the space

$$\Omega_0 = \Omega'_0 \otimes \Omega''_0, \tag{7}$$

with probabilities $\mathbb{P}(\Omega'_0)$, $\mathbb{P}(\Omega''_0) \geq 1 - Ne^{-cn}$.

In the main part of the argument $\omega'' \in \Omega''_0$ is fixed, and we investigate the random behaviour depending on $\omega' \in \Omega'_0$.

Steps I, II and III appear in some form in most (or perhaps even all) proofs of this type, and we describe them as we proceed.

Step I. Estimates for a single operator

Recall that below we take $N = n$. For $\alpha > 0$, a centrally symmetric convex body $B \subset \mathbb{R}^n$, and an operator $T \in L(\mathbb{R}^n)$, we let

$$A(\alpha, B, T) = \{\omega \in \Omega \mid \|T : K_N(\omega) \to B\| \leq \alpha\sqrt{n}\}. \tag{8}$$

Lemma 3 *Let $B \subset \mathbb{R}^n$ be a centrally symmetric convex body. For every operator $T \in L(\mathbb{R}^n)$ with $\det(T) = 1$ and every $\alpha > 0$ one has*

$$\mathbb{P}\{A(\alpha, B, T)\} \leq \left((c_0\alpha\sqrt{n})^n \frac{\mathrm{vol}\ B}{\mathrm{vol}\ B_2^n} \right)^N.$$

Proof. By Fact 1 we have

$$\begin{aligned}
\mathbb{P}\{A(\alpha,B,T)\} &= \mathbb{P}\{T(K_N(\omega)) \subset \alpha\sqrt{n}B\} \\
&\le \prod_{i=1}^{N} \mathbb{P}\{Tg_i(\omega_i) \in \alpha\sqrt{n}B\} \\
&= \prod_{i=1}^{N} \mathbb{P}\{g_i(\omega_i) \in \alpha\sqrt{n}T^{-1}B\} \\
&\le \prod_{i=1}^{N} (c_0\alpha\sqrt{n})^n \frac{\text{vol } T^{-1}B}{\text{vol } B_2^n} \\
&= \left((c_0\alpha\sqrt{n})^n \frac{\text{vol } B}{\text{vol } B_2^n} \right)^N,
\end{aligned}$$

where $c_0 > 0$ is an absolute constant. □

Lemma 4 *Let $B \subset \mathbb{R}^n$ be a centrally symmetric convex body of the form $B = \text{conv}\,\{\pm x_1, \dots, \pm x_M\}$ for $M = 2n$, and some vectors $x_i \in \mathbb{R}^n$ with the Euclidean norm $|x_i| \le 2$, for $1 \le i \le M$. Then*

$$\text{vol } B \le \left(\frac{c_1}{n} \right)^{n/2},$$

where $c_1 > 0$ is an absolute constant.

Proof. Write $B = \text{conv}\,\{y_1, \dots, y_{2M}\}$ where for every $1 \le j \le 2M$, y_j equals to $+x_i$ or $-x_i$ for some $1 \le i \le M$. For every subset σ of $\{1, 2, \dots, 2M\}$ of cardinality $n+1$, define B_σ by $B_\sigma = \text{conv}\,\{y_j \mid j \in \sigma\}$. By Caratheodory's theorem, $B = \bigcup_\sigma B_\sigma$. By Hadamard's inequality, $\text{vol } B_\sigma \le 2^n \,\text{vol } B_1^n = 4^n/n!$. So by Stirling's formula

$$\text{vol } B \le \sum_\sigma \text{vol } B_\sigma \le \binom{4M}{n+1} \frac{4^n}{n!} \le \left(\frac{c_1}{n} \right)^n,$$

where $c_1 > 0$ is an absolute constant. □

Conclusion of Step I: For every $\alpha > 0$, any fixed polytope $K_N''(\omega'') \subset 2B_2^n$, and any operator $T \in L(\mathbb{R}^n)$ with $\det T = 1$, we have

$$\mathbb{P}(A(\alpha, K_N'', T)) \le (c_2\alpha)^{n^2}, \tag{9}$$

In fact, it is sufficient to consider the sets $A(\alpha, K''_N, T)$ for a sufficiently dense net in a suitable set of operators.

Step II. Discretization, ε-nets in spaces of operators

Recall that if B_1 is a centrally symmetric convex body in $\mathbb{R}^n$, and $A \subset \mathbb{R}^n$ and $\delta > 0$ then $\mathcal{N}$ is a δ-net in A with respect to B_1 if $\mathcal{N} \subset A$ and for every $x \in A$ there is $z \in \mathcal{N}$ such that $x - z \in \delta B_1$. The following lemma is standard.

Lemma 5 *Let $B_1 \subset B_2$ be two centrally symmetric convex bodies in $\mathbb{R}^n$. For every subset $A \subset B_2$ there exists a 1-net $\mathcal{N}$ in A with respect to B_1 with cardinality $|\mathcal{N}| \leq 3^n \operatorname{vol} B_2 / \operatorname{vol} B_1$.*

Proof. Let $\mathcal{N} = \{x_1, \dots, x_M\}$ be a maximal subset of A satisfying $x_i - x_j \notin B_1$ for all $i \neq j$. The maximality implies that it is a 1-net in A. Consider the set $\bigcup_{i=1}^M (x_i + \frac{1}{2}B_1) \subset (1 + \frac{1}{2})B_2$, and note that the sets forming the union are mutually disjoint translates of $\frac{1}{2}B_1$. Thus $M(\frac{1}{2})^n \operatorname{vol} B_1 \leq (\frac{3}{2})^n \operatorname{vol} B_2$. $\square$

We shall identify, in the canonical way, operators from $L(\mathbb{R}^n)$ with $n \times n$ matrices, considered as elements of $\mathbb{R}^{n^2}$. In particular, this allows to consider the n^2-dimensional volume of any Borel set of operators. We shall consider two sets of operators (B_2^n and B_1^n below denote the unit ball in ℓ_2^n and ℓ_1^n, respectively):

$$B_{op}^n = \{T \in L(\mathbb{R}^n) \mid \|T : B_2^n \to B_2^n\| \leq 1\},$$

and, for a centrally symmetric convex body $B \subset \mathbb{R}^n$,

$$B_{op,B}^n = \{T \in L(\mathbb{R}^n) \mid \|T : B_1^n \to B\| \leq 1\}.$$

Lemma 6 *We have* $\operatorname{vol} B_{op,B}^n = (\operatorname{vol} B)^n$ *and* $\operatorname{vol} B_{op}^n \geq (c_1/n)^{n^2/2}$*, where $c_1 > 0$ is a universal constant.*

The ball $B_{op,B}^n$ has a very simple structure: it is just the Cartesian product of n copies of B, $B \times \ldots \times B$. So the formula for the first volume is obvious. The remaining lower bound is by now standard and based on a fundamental upper bound for the norm of a Gaussian matrix. For sake of completeness we briefly describe it – following [MT2] – at the end of this section, and we give a short elementary proof

We formulate the conclusion of Step II in the next corollary.

Corollary 7 *Let $B \subset \mathbb{R}^n$ be a centrally symmetric convex body and let $t > 0$ be such that $tB_2^n \subset B$. Every subset $A \subset B_{op,B}^n$ admits a t-net $\mathcal{N}$, with respect to the operator norm on ℓ_2^n with* $\mathrm{card}\,(\mathcal{N}) \le (C/t)^{n^2}(\mathrm{vol}\ B/\,\mathrm{vol}\ B_2^n)^n$*, where C is an absolute constant.*

This follows immediately from Lemmas 6 and 5, by observing that the condition $tB_2^n \subset B$ implies $tB_{op}^n \subset B_{op,B}^n$ (in turn, this observation follows by direct checking on appropriate norms of operators).

Step III. Perturbation argument, the end of the proof

Fix any $\omega'' \in \Omega_0''$ and denote $K_N''(\omega'')$ by $\widetilde{K}_N$ (recall that $\widetilde{K}_N \subset 2B_2^n$).

For any $\alpha > 0$ consider the set $A(\alpha, \widetilde{K}_N)$ of all $\omega' \in \Omega_0'$ such that there exists $T \in L(\mathbb{R}^n)$, with $\det T = 1$ such that $\|T : K_N'(\omega') \to \widetilde{K}_N\| \le \alpha\sqrt{n}$. This is our "ultimately bad" set. Any polytope K_N' associated with this set admits operators of small operator norms from K_N' to $\widetilde{K}_N$. It is clear that we need to remove this set from our considerations.

We have

$$A(\alpha, \widetilde{K}_N) = \bigcup_T A(\alpha, \widetilde{K}_N, T), \tag{10}$$

where the union runs over all $T \in L(\mathbb{R}^n)$ with $\det T = 1$.

Lemma 8 *For sufficiently small $\alpha > 0$ one has, for every $n \ge 1$, $\mathbb{P}(A(\alpha, \widetilde{K}_N)) \le 2^{-n^2}$.*

Proof. Let $\mathcal{N}$ be an α-net with minimal cardinality in the set of all operators $T \in L(\mathbb{R}^n)$, with $\det T = 1$ such that $\|T : B_1^n \to \widetilde{K}_N\| \le \alpha\sqrt{n}$, with respect to the operator norm on ℓ_2^n. By Corollary 7, $|\mathcal{N}| \le C_1^{n^2}$, where $C_1 > 1$ is a universal constant.

We are going to prove that

$$A(\alpha, \widetilde{K}_N) \subset \bigcup_{T \in \mathcal{N}} A(3\alpha, \widetilde{K}_N, T). \tag{11}$$

Having proved this, by (9) we get

$$\mathbb{P}(\bigcup_{T \in \mathcal{N}} A(3\alpha, \widetilde{K}_N, T)) \le C_1^{n^2}(3c_2\alpha)^{n^2}.$$

In turn, for sufficiently small $\alpha > 0$ this is bounded above by $(1/2)^{n^2}$, concluding the proof.

Comments before proving (11). This is the second crucial point of the argument. The set defined in (10) is described by a certain condition satified for *all* operators T. By comparison, in (11) we consider only a *subset* of operators T, but this is possible because we weaken the appropriate condition (from $\leq \alpha\sqrt{n}$ to $\leq 3\alpha\sqrt{n}$)

Now we prove (11). Fix $\omega' \in A(\alpha, \widetilde{K}_N)$. Let T be an operator with $\det T = 1$ such that $\|T : K'_N(\omega') \to \widetilde{K}_N\| \leq \alpha\sqrt{n}$. Since K'_N contains the ball B_1^n then $T \in (\alpha\sqrt{n})B^n_{op,\widetilde{K}_N}$.

Pick $T_0 \in \mathcal{N}$ with $\|T - T_0 : B_2^n \to B_2^n\| \leq \alpha$. Since

$$n^{-1/2}B_2^n \subset B_1^n \subset K'_N(\omega') \qquad \text{and} \qquad \widetilde{K}_N \subset 2B_2^n,$$

we get

$$\begin{aligned} \|T_0 : K'_N(\omega') \to \widetilde{K}_N\| &\leq \|T : K'_N(\omega') \to \widetilde{K}_N\| + \|T_0 - T : K'_N(\omega') \to \widetilde{K}_N\| \\ &\leq \alpha\sqrt{n} + 2\sqrt{n}\|T_0 - T : B_2^n \to B_2^n\| \leq 3\alpha\sqrt{n}. \end{aligned}$$

Thus $\omega' \in A(3\alpha, \widetilde{K}_N, T_0)$, with $T_0 \in \mathcal{N}$. □

Proof of Theorem 2 Fix $\alpha > 0$ satisfying Lemma 8. Denote by $\mathcal{Q}$ the set of all pairs $(\omega', \omega'') \in \Omega'_0 \otimes \Omega''_0$ such that for all $T \in L(\mathbb{R}^n)$ with $\det T = 1$ one has $\|T : K'_N(\omega') \to K''_N(\omega'')\| > \alpha\sqrt{n}$. Set $\mathcal{T} = \{(\omega', \omega'') \mid (\omega'', \omega') \in \mathcal{Q}\}$. Using the Fubini theorem for the complement of $\mathcal{Q}$, by (7) and Lemma 8, we get

$$\mathbb{P} \times \mathbb{P}(\mathcal{Q}) = \mathbb{P} \times \mathbb{P}(\mathcal{T}) \geq 1 - N\exp(-cn) - 2^{-n^2}.$$

Hence $\mathbb{P} \times \mathbb{P}(\mathcal{Q} \cap \mathcal{T}) \geq 1 - 2N\exp(-cn) + 2^{-n^2})$.

To complete the proof note that $\mathrm{d}(K'_N(\omega'), K''_N(\omega'')) > \alpha^2 n$ for $(\omega', \omega'') \in \mathcal{Q} \cap \mathcal{T}$. Indeed, let $T \in L(\mathbb{R}^n)$ be an arbitrary isomorphism. By multiplying T by a suitable constant we may assume that $\det T = 1$ and $\det T^{-1} = 1$. Hence each of the norms $\|T : K'_N(\omega') \to K''_N(\omega'')\|$ and $\|T^{-1} : K''_N(\omega'') \to K'_N(\omega')\|$ is larger than $\alpha\sqrt{n}$. □

At the end of the section we give the proof of Lemma 6.

Proof of Lemma 6 The argument for the lower bound for the volume of B^n_{op} is based on well known properties of Gaussian matrices. Let $G(\omega)$ be an $n \times n$ Gaussian matrix whose columns are independent normalized Gaussian vectors in $\mathbb{R}^n$. First recall the tail behaviour of a Gaussian matrix: for every $a > 0$ we have

$$\mathbb{P}\{\omega \mid \|G(\omega) : \ell_2^n \to \ell_2^n\| \geq a\} \leq (6\sqrt{2}e^{-a^2/16})^n. \tag{12}$$

(Although these are definitely not the best constants, but they are rather easy to prove and hence convenient for us to use (see e.g., [MT2] for the proof based on the same ingredients we used here).

Now observe that $n^{-1/2}G$ is a normalized Gaussian vector in $\mathbb{R}^{n^2}$. Therefore, applying Fact 1 for the set $B = 8n^{-1/2}B_{op}^n \subset \mathbb{R}^{n^2}$ we get

$$\left(\mathrm{vol}\ B/\,\mathrm{vol}\ B_2^{n^2}\right) \geq (n/8^2e)^{n^2/2}\mathbb{P}\big\{\|G : \ell_2^n \to \ell_2^n\| \leq 8\big\}.$$

By (12), probability of the set above is larger than or equal to 1/2, hence, using the formula for the volume vol $B_2^{n^2}$ we get

$$\mathrm{vol}\ B_{op}^n \geq (1/2)(n/8^2e)^{n^2/2}\,\mathrm{vol}\ B_2^n \geq (c'/n)^{n^2/2},$$

where $c' > 0$ is a universal constant.

Finally, for the convenience of a non-specialist reader, we sketch the argument for (12).

Fix an arbitrary $a' > 0$. Since matrices G and GU have the same distribution for any fixed orthogonal matrix U, then by (2) we have, for every $x \in S^{n-1}$,

$$\mathbb{P}\{\|G(\omega)x\|_2 > a'\} = \mathbb{P}\{\|G(\omega)e_1\|_2 > a'\} \leq (\sqrt{2}e^{-a'^2/4})^n.$$

By Lemma 5, the unit sphere S^{n-1} admits an 1/2-net $\mathcal{N}$ with respect to $\|\cdot\|_2$ with cardinality not greater than 6^n. Set $A = \{\omega \in \Omega \mid \|G(\omega)x\|_2 \leq a'$ for all $x \in \mathcal{N}\}$. Then $\mathbb{P}(A) \geq 1 - (6\sqrt{2}e^{-a'^2/4})^n$. An easy approximation argument shows that $\|G(\omega) : \ell_2^n \to \ell_2^n\| \leq 2a'$ for every $\omega \in A$. Thus the conclusion follows from the estimate for $\mathbb{P}(A)$, setting $a' = a/2$. □

2 Banach–Mazur distances between random projections of convex bodies

Gluskin's random polytopes K_N in $\mathbb{R}^n$ investigated in the previous section are obtained as linear images of the higher dimensional cross-polytope $B_1^{N'}$

in $\mathbb{R}^{N'}$, where $N' = n + N = 2n$. Indeed, (6) defines K_N via certain matrix, a part of which is a random Gaussian matrix. This matrix can be actually made into a full $n \times N'$ random Gaussian matrix, although the proofs are more technically complicated. Some results of this type were also proved by various authors for *Gaussian projections* of the balls $B_p^{N'}$ for $1 < p < 2$. This in turn leads to similar result about Banach-Mazur distance between *random orthogonal projections* $P_H(B_1^{N'})$ or $P_H(B_p^{N'})$ (One should however note that passing from Gaussian results to analogous results for projections is not purely formal and sometimes a rather delicate matter.)

It is very interesting that similar results are true for random projections of an *arbitrary* centrally symmetric convex body $K \subset \mathbb{R}^N$. As an example of just one such result, the diameter of a family of random n-dimensional orthogonal projections of such a body $K \subset \mathbb{R}^N$ (for any $1 \leq n < N$ was studied in [MT1] and it was shown that this diameter is larger than or equal to the square of the Euclidean distances of random k-dimensional projections of the body (where $k = (1/2 - \varepsilon)n$, for any $\varepsilon > 0$). More precisely,

Theorem 9 *For $1 \leq n \leq N$ denote by $G_{N,n}$ the Grassmann manifold of n-dimensional subspaces of $\mathbb{R}^N$ with the normalized Haar measure $\mu_{N,n}$; and for a subspace $H \subset \mathbb{R}^N$, by P_H denote the orthogonal projection onto H. Let K be a symmetric convex body in $\mathbb{R}^N$ such that the Euclidean unit ball B_2^N is the ellipsoid of minimal volume containing K, let $0 < \lambda < 1$ and assume that $n \leq \lambda N$. Then*

$$\int_{G_{N,n}} \int_{G_{N,n}} d\left(P_{H_1}(K), P_{H_2}(K)\right)\, d\mu_{N,n}(H_1)\, d\mu_{N,n}(H_2) \geq c \left(\int_{G_{N,m}} d\left(P_H(K), B_2^m\right)\, d\mu_{N,m}(H) \right)^2, \tag{13}$$

where $c = c(\lambda) > 0$ depends on λ only, and $m = [2n/5]$, say.

In particular this shows that (up to the drop in the dimension) the Banach–Mazur distance between random n-dimensional projections of an arbitrary symmetric convex body $K \subset \mathbb{R}^n$ is of maximal order allowed by a given Euclidean distance of random projections of K of a slightly smaller dimension. We should also observe that the drop of dimension is necessary and the formula is in a certain sense optimal.

Most of results in this and related directions proved before 2000 are described in the survey paper [MT2] and references therein. Theorem 9 is a main result of [MT1]; further developments and applications of the theory can be found in later papers by various authors.

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