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Absolutely summing operators in $\mathscr{L}_p$-spaces and their applications

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1. INTRODUCTION

The main purpose of the present paper is to give a new presentation as well as new applications of the results contained in Grothendieck's paper [17]. In this remarkable paper Grothendieck outlined the theory of tensor products of Banach spaces. The climax of this paper was a theorem called by Grothendieck "the fundamental theorem of the metric theory of tensor products". This theorem is equivalent to the following assertion:

Let $\{a_{i,j}\}_{i,j=1}^{n}$ be a finite matrix of real numbers such that

$$\Big|\sum_{i,j=1}^{n} a_{i,j}t_i s_j\Big| \leqslant 1$$

whenever $|t_i| \leqslant 1$, $|s_j| \leqslant 1$. Then for every set of unit vectors $\{x_i\}_{i=1}^{n}$ and $\{y_j\}_{j=1}^{n}$ in a Hilbert space

$$\Big|\sum_{i,j} a_{i,j}(x_i, y_j)\Big| \leqslant K,$$

where K is an absolute constant and $(\cdot,\cdot)$ denotes the inner product in the Hilbert space.

This inequality, as well as many of its applications, are meaningful and interesting also outside the framework of tensor product theory. Though the theory of tensor products constructed in Grothendieck's paper has its intrisic beauty we feel that the results of Grothendieck and their corollaries can be more clearly presented without the use of tensor products. The paper of Grothendieck is quite hard to read [1] and its results are not generally known even to experts in Banach space theory. In fact, by using these results some problems which were posed by various authors in the last decade can be easily solved. All these considerations persuaded us to write this paper in its present form. We do not use here the notion of tensor products.

(1) An elegant exposition of the introductory part of [17] can be found in [56].

In Section 2 we present a proof of the inequality mentioned above and of its immediate consequences. The proof we present is just a reformation of the argument of Grothendieck. The proof is elementary and no knowledge of functional analysis is needed for its understanding.

Section 3 is devoted to functional analytic preliminaries. In particular, we introduce in it the class of $\mathscr{L}_p$-spaces, $1 \leqslant p \leqslant \infty$. These are Banach spaces whose finite-dimensional subspaces are the "same" as those of an $L_p(\mu)$ space for some measure μ. These spaces are introduced since most of the results proved in the present paper depend not on the whole Banach space but rather on the structure of its finite-dimensional subspaces. We present also the notion of p-absolutely summing operators ($1 \leqslant p < \infty$) which is due to Pietsch [51] (cf. Saphar [52], [58] for $p=2$) and which for $p = 1$ goes back to Grothendieck. The applications of the inequality of Section 2 to the theory of Banach spaces are made through the use of this notion of p absolutely summing operators. This is done in Section 4. We prove there that every operator from an $\mathscr{L}_1$-space to a Hilbert space is 1-absolutely summing and that this property characterizes, in a certain sense, $\mathscr{L}_1$ and Hilbert spaces respectively. As a corollary it follows that the inequality of Section 2 (which was stated above) characterizes Banach spaces which are isomorphic to Hilbert spaces. It also is shown in Section 4 that every operator from an $\mathscr{L}_\infty$ space to an $\mathscr{L}_p$ space, $1 \leqslant p \leqslant 2$, is 2-absolutely summing.

The results of Section 4 are used in Section 5 for obtaining factorization theorems for certain classes of operators. The main result here is that every linear operator T from an $\mathscr{L}_p$-space X into an $\mathscr{L}_r$-space Y where $p > 2 > r$ can be represented as $T = T_1 T_0$, where T_0 is a linear operator from X into a suitable Hilbert space H and T_1 is a linear operator from H into Y.

Section 6 is devoted to various applications of the preceding results. One application is the following: In the spaces l_1 and c_0 all normalized unconditional bases are equivalent to the usual unit basis. The space l_1 (resp. c_0) is the only complemented subspace of an $\mathscr{L}_1$ (resp. $\mathscr{L}_\infty$) space which has an unconditional basis. A qualitative version of this result gives a new connection between the projection and symmetry constants of a finite-dimensional space X and its distance from the space l_∞^n (with $n = \dim X$).

The results in Sections 4 and 5 concerning operators defined on $\mathscr{L}_p$-spaces provide a tool for proving that certain subspaces of $\mathscr{L}_p$-spaces are not complemented subspaces. We show in Section 6 how to use this tool in order to give a new proof to the result of D. J. Newman that the Hardy space H_1 is not a complemented subspace of $L_1(\mu)$ (where μ is the Haar measure on the circle).

Another application which is presented in Section 6 is Grothendieck's characterization of a Hilbert space as the only Banach space which is isomorphic to a subspace of an $\mathscr{L}_1$-space and to a quotient space of an $\mathscr{L}_\infty$-space. We also present in this section several characterizations, due essentially to Grothendieck, of Hilbert-Schmidt and trace-class operators in a Hilbert space.

Section 7 is devoted to a study of subspaces of $L_p(\mu)$-spaces. This study clarifies somewhat the relation between general $\mathscr{L}_p$-spaces and $L_p(\mu)$-spaces. We show in particular that every $\mathscr{L}_p$-space, $1 < p < \infty$, is isomorphic to a complemented subspace of an $L_p(\mu)$-space for a suitable measure μ. Examples, given in Section 8, show that this is no longer true if $p = 1$ or ∞ and that unless $p = 2$ the class of $\mathscr{L}_p$-spaces properly includes the class of spaces isomorphic to $L_p(\mu)$-spaces. In Section 7 it is also shown that by combining known results it is now possible to give a complete solution to the problem of the linear dimension of $L_p(\mu)$-spaces (cf. Banach [2]).

The last section contains, besides the examples mentioned above, some open problems and various additional remarks and results. The main result in this section is the proof of the existence of a "universal" non-weakly compact operator.

Notation and terminology are given in Section 3. Let us only mention here that unless stated otherwise we consider only spaces over the reals though all the results and proofs carry over to the complex case.

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2. THE BASIC INEQUALITY

In this section we present the inequalities which form the basis of most of the proofs in the following sections. These inequalities are of interest in themselves and may be of use also to mathematicians who are not working in Banach space theory.

Let $S = S^n = \{x \in E^n; \|x\| = 1\}$ denote the $(n-1)$-dimensional sphere in the n-dimensional real Euclidean space $E = E^n$. Let m be the rotation invariant Borel measure on S normalized so that $m(S) = 1$. Let

$$(x, y) = \sum_{i=1}^{n} x^i y^i$$

denote the usual inner product of the vectors $x = (x^1, \ldots, x^n)$ and $y = (y^1, \ldots, y^n)$ in E^n. For real t let $\operatorname{sign} t = t/|t|$ if $t \neq 0$ and $\operatorname{sign} 0 = 0$.

LEMMA 2.1. *Let $x, y \in S^n$; then*

$$\int_{S^n} \operatorname{sign}(x, u) \operatorname{sign}(y, u)\, dm(u) = 1 - \frac{2}{\pi}\theta(x, y), \tag{2.1}$$

where $\theta = \theta(x, y)$ is the unique number satisfying $\cos\theta = (x, y)$ and $0 \leqslant \theta \leqslant \pi$ (i.e. θ is the angle between x and y).

Proof. We choose the basis in E^n in such a way that $x = (1, 0, \ldots, 0)$ and $y = (\cos\theta, \sin\theta, 0, 0, \ldots, 0)$. Let g be a bounded measurable function on S^n. Using polar coordinates $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_{n-1})$ we express the integral $\int_{S^n} g(u)dm(u)$ by the $(n-1)$-dimensional Lebesgue integral. We have the relation

$$\int_{S^n} g(u)\,dm(u) = |S^n|^{-1} \int_{I^{n-1}} g\big(u(\varphi)\big) J(\varphi)\,d(\varphi),$$

where

$$u(\varphi) = \big(u^1(\varphi), u^2(\varphi), \ldots, u^n(\varphi)\big), \qquad u^1(\varphi) = \prod_{i=1}^{n-1} \sin\varphi_i,$$

$$u^k(\varphi) = \cos\varphi_{k-1} \prod_{i=k}^{n-1} \sin\varphi_i \quad \text{for} \quad k = 2, 3, \ldots, n-1,$$

$$u^n(\varphi) = \cos\varphi_{n-1},$$

$$I^{n-1} = \{\varphi\colon 0 \leqslant \varphi_1 < 2\pi;\ 0 \leqslant \varphi_i \leqslant \pi \ \text{for}\ i = 2, 3, \ldots, n-1\},$$

$$J(\varphi) = \prod_{i=2}^{n-1} (\sin\varphi_i)^{i-1},$$

$$|S^n| = \int_{I^{n-1}} J(\varphi)\,d(\varphi) = 2\pi \prod_{i=2}^{n-1} \int_0^{\pi} (\sin\varphi_i)^{i-1}\,d\varphi_i.$$

Let $h(u) = (x, u)(y, u) = u^1(u^1\cos\theta + u^2\sin\theta)$. Then

$$h\big(u(\varphi)\big) = \Big[\prod_{i=2}^{n-1} \sin\varphi_i\Big]^2 \sin\varphi_1(\sin\varphi_1\cos\theta + \cos\varphi_1\sin\theta).$$

Hence, for $g(u) = \operatorname{sign}[h(u)]$, we get

$$g\big(u(\varphi)\big) = \operatorname{sign}[\sin\varphi_1 \sin(\varphi_1 + \theta)] = f(\varphi_1, \theta).$$

Clearly, $f(\varphi_1, \theta)$ is equal to $+1$ on the intervals $(0; \pi - \theta)$ and $(\pi; 2\pi - \theta)$, and is equal to -1 on the intervals $(\pi - \theta; \pi)$ and $(2\pi - \theta; 2\pi)$. Thus

$$\int_{S^n} g(u)\,dm(u) = |S^n|^{-1} \int_{I^{n-1}} f(\varphi_1, \theta) J(\varphi)\,d(\varphi)$$

$$= |S^n|^{-1} \int_0^{2\pi} f(\varphi_1, \theta)\,d\varphi_1 \prod_{i=2}^{n-1} \int_0^{\pi} (\sin\varphi_i)^{i-1}\,d\varphi_i$$

$$= (2\pi)^{-1} \int_0^{2\pi} f(\varphi_1, \theta)\,d\varphi_1 = 1 - 2\theta/\pi.$$

This completes the proof.

We are now ready for the proof of the main result:

THEOREM 2.1. *Let $\{a_{i,j}\}_{i,j=1,2,\ldots,N}$ be a real-valued matrix and let M be a positive number such that*

$$\Big|\sum_{i,j=1}^{N} a_{i,j} t_i s_j\Big| \leqslant M \tag{2.4}$$

for every real $\{t_i\}_{i=1}^N$ and $\{s_j\}_{j=1}^N$ satisfying $|t_i| \leqslant 1$ and $|s_j| \leqslant 1$. Then for arbitrary vectors $\{x_i\}_{i=1}^N$ and $\{y_j\}_{j=1}^N$ in a real inner product space H

$$\Big|\sum_{i,j=1}^{N} a_{i,j}(x_i, y_j)\Big| \leqslant K_G M \sup_i \|x_i\| \sup \|y_j\|, \tag{2.5}$$

where K_G is the Grothendieck universal constant $(K_G \leqslant \sinh\pi/2 = (e^{\pi/2} - e^{-\pi/2})/2)$.

Proof. Let us first make some observations.

1° If a matrix $\{a_{i,j}\}$ satisfies (2.4), then for arbitrary real numbers c_i' and c_j'' $(i, j = 1, 2, \ldots, N)$ the matrix $\{a_{i,j}'\}$ with $a_{i,j}' = c_i' a_{i,j} c_j''$ for $i, j = 1, \ldots, N$ satisfies (2.4) with the constant

$$M' = M \sup_i |c_i'| \sup_j |c_j''|.$$

2° Since every $2N$ vectors in H belong to some $2N$-dimensional linear subspace of H which is isometric to E^{2N}, we may assume without loss of generality that $\{x_i\}_{i=1}^N$ and $\{y_j\}_{j=1}^N$ belong to E^{2N}. From observation 1° and a standard homogeneity argument it follows that we may assume also that $\|x_i\| = \|y_j\| = 1$ for every i and j.

For an arbitrary $u \in S^{2N}$ we define $t_i(u) = \operatorname{sign}(u, x_i)$ and $s_j(u) = \operatorname{sign}(u, y_j)$, $i, j = 1, \ldots, N$. By (2.4)

$$-M \leqslant \sum_{i,j=1}^{N} a_{i,j} t_i(u) s_j(u) \leqslant M \quad \text{for} \quad u \in S^{2N}.$$

Hence by integrating over S^{2N} with respect to the normalized rotation invariant measure we get, by formula (2.1),

$$-\frac{\pi}{2} M \leqslant \sum_{i,j=1}^{N} a_{i,j} \Big(\frac{\pi}{2} - \theta(x_i, y_j)\Big) \leqslant \frac{\pi}{2} M.$$

Let us put $a_{i,j}^{(1)} = a_{i,j}\big(\pi/2 - \theta(x_i, y_j)\big)$ for $i, j = 1, 2, \ldots, N$. It follows easily from observation 1° that the matrix $(a_{i,j}^{(1)})$ satisfies (2.4) if we replace M by $\pi M/2$. Hence, by repeating the averaging argument we get

$$-\Big(\frac{\pi}{2}\Big)^2 M \leqslant \sum_{i,j=1}^{N} a_{i,j}^{(1)} \Big(\frac{\pi}{2} - \theta(x_i, y_j)\Big) = \sum_{i,j=1}^{N} a_{i,j} \Big(\frac{\pi}{2} - \theta(x_i, y_j)\Big)^2 \leqslant \Big(\frac{\pi}{2}\Big)^2 M.$$

In this manner we obtain inductively

$$(2.6)\qquad -\left(\frac{\pi}{2}\right)^n M \leqslant \sum_{i,j=1}^{N} a_{i,j}\left(\frac{\pi}{2}-\theta(x_i, y_j)\right)^n \leqslant \left(\frac{\pi}{2}\right)^n M, \qquad n=1,2,\ldots$$

Since

$$(x_i, y_j) = \cos\theta(x_i, y_j) = \sin\left(\frac{\pi}{2}-\theta(x_i, y_j)\right)$$

$$= \sum_{n=0}^{\infty}(-1)^n\left(\frac{\pi}{2}-\theta(x_i, y_j)\right)^{2n+1}/(2n+1)!,$$

inequality (2.6) implies that

$$\left|\sum_{i,j=1}^{N} a_{i,j}(x_i, y_j)\right| \leqslant M\sum_{n=0}^{\infty}\left(\frac{\pi}{2}\right)^{2n+1}\frac{1}{(2n+1)!} = M\sinh\frac{\pi}{2}$$

and this concludes the proof of the theorem.

COROLLARY 1. *Let $\{a_{i,j}\}$ be a real-valued matrix for which* (2.4) *holds. Then for arbitrary vectors $\{x_i\}_{i=1}^N$ in an inner product space H*

$$(2.7)\qquad \sum_{j=1}^{N}\left\|\sum_{i=1}^{N} a_{i,j}x_i\right\| \leqslant K_G M\sup_i\|x_i\|.$$

Proof. Choose for $j=1,\ldots,N$ vectors $y_j\in H$ such that $\|y_j\|=1$ and

$$\left(\sum_{i=1}^{N} a_{i,j}x_i, y_j\right) = \left\|\sum_{i=1}^{N} a_{i,j}x_i\right\|.$$

By using these x_i and y_j in (2.5) we get (2.7).

COROLLARY 2. *Let $\{a_{i,j}\}_{i,j=1,2,\ldots}$ be an infinite real matrix and let M be a positive constant such that*

$$(2.8)\qquad \left|\sum_{i,j=1}^{N} a_{i,j}t_i s_j\right| \leqslant M \quad \text{for} \quad |t_i|\leqslant 1, |s_j|\leqslant 1, \quad i,j,N=1,2,\ldots$$

Then for an arbitrary real matrix $\{x_{k,i}\}$ such that for some $C>0$

$$(2.9)\qquad \left(\sum_k x_{k,i}^2\right)^{1/2} \leqslant C \quad \text{for} \quad i=1,2,\ldots$$

the following inequalities hold:

$$(2.10)\qquad \sum_j\left(\sum_k\left(\sum_i x_{k,i}a_{i,j}\right)^2\right)^{1/2} \leqslant K_G CM,$$

"general Littlewood inequality", and

$$(2.11)\qquad \left(\sum_k\left(\sum_j\left|\sum_i x_{k,i}a_{i,j}\right|\right)^2\right)^{1/2} \leqslant K_G CM,$$

"general Orlicz inequality."

Proof. Observe first that (2.8) implies that

$$\sum_i |a_{i,j}| \leqslant M \qquad (j=1,2,\ldots).$$

Since, by (2.9), $|x_{i,k}|\leqslant C$ for every i and k, the series $\sum_i x_{k,i}a_{i,j}$ is absolutely convergent for $k,j=1,2,\ldots$ Therefore, since the sums over k and j in (2.10) and (2.11) have non-negative terms, it is enough to restrict our attention to the case where $\{x_{k,i}\}$ is a matrix with an arbitrary but finite number of elements different from zero (we pass to the general case by a standard limit procedure). Hence in the sequel we shall assume that each of the sums appearing in (2.9), (2.10) or (2.11) has exactly N terms.

Let $x_i=(x_{k,i})$ denote the i-th column of the matrix $\{x_{k,i}\}$ $(i=1,\ldots,N)$. We consider the x_i as vectors in the N-dimensional Euclidean space E^N. Then (2.9) means that $\|x_i\|\leqslant C$ for every i, and thus (2.10) is just a reformulation of (2.7).

Inequality (2.11) is an immediate consequence of (2.10). In fact, let

$$b_{j,k} = \left|\sum_i x_{k,i}a_{i,j}\right|.$$

By the triangle inequality for the vectors $b_j=(b_{j,k})$, $j=1,\ldots,N$, in E^N

$$\left(\sum_k\left(\sum_j b_{j,k}\right)^2\right)^{1/2} \leqslant \sum_j\left(\sum_k b_{j,k}^2\right)^{1/2},$$

i.e. the expression in the left-hand side of (2.11) is not larger than the expression in the left-hand side of (2.10).

Remark. If $x_{k,i}=\delta_i^k(=1$ for $i=k$ and $=0$ otherwise), (2.10) reduces to the inequality

$$\sum_j\left(\sum_i a_{i,j}^2\right)^{1/2} \leqslant K_G M.$$

This inequality (with a better constant, $\sqrt{3}$ instead of K_G) is due to Littlewood [38] (see also [50], p. 39, and [49]). For the same choice of $x_{k,i}$ formula (2.11) reduces to the inequality

$$\left(\sum_i\left(\sum_j|a_{i,j}|\right)^2\right)^{1/2} \leqslant K_G M.$$

This inequality was obtained by Orlicz in [42]. As in the proof of Theorem 2.1, the inequalities of Littlewood and Orlicz were obtained from (2.8) by using an averaging procedure. It would be of some interest to know the best possible value for K_G as well as the best constant in the inequalities of Littlewood and Orlicz (i.e. inequalities (2.10) and (2.11) with $x_{k,i} = \delta_i^k$). Grothendieck proves in [17] that $K_G \geqslant \pi/2$.

Let us finally note that if we consider also complex-valued matrices $\{a_{i,j}\}$ for which (2.4) holds, then (2.5) will be valid (in complex or real Hilbert spaces) if K_G is replaced by $2K_G$. In order to see this we have only to take the real and imaginary parts of the matrix $\{a_{i,j}\}$ and to use inequality (2.7) which is equivalent to (2.5).