Khinchine's Inequality

We consider functions $r_1, r_2, \dots$ on $[0,1)$

defined by

$$r_1(t) = \begin{cases} +1 & \text{on } [0, 1/2) \\ -1 & \text{on } [\frac{1}{2}, 1) \end{cases}$$

$$r_2(t) = \begin{cases} +1 & \text{on } [0, \frac{1}{4}) \cup [1/2, \frac{3}{4}) \\ -1 & \text{on } [\frac{1}{4}, \frac{1}{2}) \cup [\frac{3}{4}, 1) \end{cases} \qquad \text{etc}$$

for the nth function we divide $[0,1]$ on 2^n subintervals, each of length 2^{-n}, and define $r_n(t)$ to be equal $+1$ or -1 on successive subintervals.

We shall consider products of Rademacher functions:

$$\int_0^1 r_1(t)^{i_1} r_2(t)^{i_2} \dots r_n(t)^{i_n} \, dt$$

This $= 1$ if all powers $i_1, i_2, \dots, i_n$ are <u>even</u>

otherwise $= 0$

Consider $L_p = L_p(0,1)$ with Lebesgue measure

$$\|f\|_p = \left(\int_0^1 |f(t)|^p \, dt \right)^{1/p}$$

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where $\|c\|_2$ is the ℓ_2-norm of the vector $c = (c_1, \ldots, c_n, 0, 0, \ldots)$. It is a surprising fact that the L_p-norm of $\sum_{j=1}^n c_j r_j$ is equivalent to the ℓ_2-norm of the vector c for every value of p.

Theorem 2.24 (Khinchine's Inequality). *Let r_j be the Rademacher functions. For each $p \in [1, \infty)$ there are positive constants A_p and B_p such that*

$$A_p\|c\|_2 \le \left\| \sum_{j=1}^n c_j r_j \right\|_p \le B_p\|c\|_2$$

for every finite sequence $c = (c_1, \ldots, c_n)$ of scalars.

Proof. We have already seen that $\|\sum_{j=1}^n c_j r_j\|_2 = \|c\|_2$ and so we may take $A_2 = B_2 = 1$. Now the norm of $L_p[0,1]$ is an increasing function of p and so it suffices to prove the B_p inequality for $p > 2$ and the A_p inequality for $p < 2$.

Step 1: The B_p inequality for $p > 2$ and $\mathbf{K} = \mathbf{R}$.
By the monotonicity of the L_p-norms, it is enough to prove the inequality when p is an even positive integer. Accordingly, let $p = 2m$. Using the Multinomial Theorem, we have

$$\begin{aligned}\left\| \sum_{j=1}^n c_j r_j \right\|_p^p &= \int_0^1 \left(\sum_{j=1}^n c_j r_j(t) \right)^{2m} dt \\ &= \sum_{k_1+\cdots+k_n=2m} \frac{(2m)!}{k_1! \ldots k_n!} c_1^{k_1} \ldots c_n^{k_n} \int_0^1 r_1^{k_1}(t) \ldots r_n^{k_n}(t)\, dt\,.\end{aligned}$$

The integral is zero unless all the k_j are even, say $k_j = 2m_j$. Now $k_1 + \cdots + k_n = 2m$ if and only if $m_1 + \cdots + m_n = m$ and so

$$\begin{aligned}\left\| \sum_{j=1}^n c_j r_j \right\|_p^p &= \sum_{m_1+\cdots+m_n=m} \frac{(2m)!}{(2m_1)! \ldots (2m_n)!} c_1^{2m_1} \ldots c_n^{2m_n} \\ &\le (2m)! \sum_{m_1+\cdots+m_n=m} (c_1^2)_1^m \ldots (c_n^2)_n^m \le (2m)! \left(\sum_{j=1}^n c_j^2 \right)^m .\end{aligned}$$

Therefore

$$\left\| \sum_{j=1}^n c_j r_j \right\|_p \le (p!)^{1/p} \|c\|_2\,,$$

when p is an even integer.

Step 2: The B_p inequality for $p > 2$ and $\mathbf{K} = \mathbf{C}$.
If $c_j = a_j + ib_j$ are complex numbers, then

$$\begin{aligned}\left\| \sum_{j=1}^n c_j r_j \right\|_p &\le \left\| \sum_{j=1}^n a_j r_j \right\|_p + \left\| \sum_{j=1}^n b_j r_j \right\|_p \le B_p(\|a\|_2 + \|b\|_2) \\ &\le \sqrt{2} B_p (\|a\|_2^2 + \|b\|_2^2)^{1/2} = \sqrt{2} B_p \|c\|_2 .\end{aligned}$$

Step 3: The A_p inequality for $1 \le p < 2$.
By the monotonicity of the L_p-norms, it suffices to prove the inequality in the case $p = 1$. Using Hölder's inequality with the conjugate exponents $3/2$ and 3 and the B_p inequality for $p = 4$, we have

$$\begin{aligned}\|c\|_2^2 = \int_0^1 \left| \sum_{j=1}^n c_j r_j \right|^2 dt &= \int_0^1 \left| \sum_{j=1}^n c_j r_j \right|^{2/3} \left| \sum_{j=1}^n c_j r_j \right|^{4/3} dt \\ &\le \left(\int_0^1 \left| \sum_{j=1}^n c_j r_j \right| dt \right)^{2/3} \left(\int_0^1 \left| \sum_{j=1}^n c_j r_j \right|^4 dt \right)^{1/3} \\ &\le \left\| \sum_{j=1}^n c_j r_j \right\|_1^{2/3} B_4^{4/3} \|c\|_2^{4/3} .\end{aligned}$$

It follows that

$$\left\| \sum_{j=1}^n c_j r_j \right\|_1 \ge B_4^{-2} \|c\|_2$$

and so we have proved the A_1 inequality with $A_1 = B_4^{-2}$. □

We should point out that the proof we have given here does not by any means give the best values for the constants A_p and B_p.

Khinchine's inequality shows that the closed subspace of $L_p[0,1]$ generated by the Rademacher functions is isomorphic to ℓ_2. For values of p greater than 1 we can say a little more about this subspace:

Theorem 2.25. *Let $1 \le p < \infty$. The closed subspace R_p of $L_p[0,1]$ generated by the Rademacher functions is isomorphic to ℓ_2. If $p > 1$ then R_p is complemented in $L_p[0,1]$.*

Proof. It follows from Khinchine's inequality that the mapping $\sum_{j=1}^n c_j e_j \mapsto \sum_{j=1}^n c_j r_j$ extends to an isomorphism from ℓ_2 onto R_p.

Now suppose that $p \ge 2$. Then $L_p[0,1] \subset L_2[0,1]$ and so, for every f in $L_p[0,1]$, the "Rademacher coefficients",

$$\langle f, r_n \rangle = \int_0^1 f(t) r_n(t)\, dt\,,$$

of f are square summable. We claim that the required projection of $L_p[0,1]$ onto R_p is given by the formula

$$P_p f = \sum_{n=1}^\infty \langle f, r_n \rangle r_n\,.$$

First, we use the Cauchy criterion to show that this series converges in $L_p[0,1]$. We have