# Solutions of Additional Exercises.

**Problem 1.** Solve Problem 2 from Homework Assignment # 5 with the condition "$\mu\{|f| > t\} \leq 1/t$ for every $t > 1$" instead of "$\mu\{|f| > t\} \leq 1/t^2$ for every $t > 1$."

**Solution.** Nothing can be said about integrability of $f$. As in Problem 2, $f$ can be integrable (just take $f(x) = 0$ for every $x$), but one can construct an example of non-integrable functions. Indeed, let $E = [0, 1]$ and $\mu$ is the Lebesgue measure (then $\mu(E) < \infty$). Let $f(x) = 1/x$ for $x > 0$ and $f(0) = 0$. Then

$$\mu(\{x \in E \mid |f(x)| > t\}) = \mu((0, 1/t)) = 1/t,$$

but

$$\int_E f \, d\mu = \int_0^1 t^{-1} \, dt = \infty.$$

Of course the condition "$\mu(E) < \infty$" is not important: if one can construct an example with $\mu(E) < \infty$ and a function $f$ then the same example is easily adjusted for the case of infinite measure – just fix $a \notin E$ and consider $F = E \cup \{a\}$ with measure $\nu$, which is the same as $\mu$ on measurable subsets of $E$ and which is infinity at $\{a\}$. Extend the function $f$ from $E$ to $F$ by defining $f(a) = 0$. Clearly, $\nu(F) = \infty$ and

$$\int_E f \, d\mu = \int_F f \, d\nu.$$

□

**Problem 2.** Let $X = (X, S)$ be a measurable space, $\mu$ and $\nu$ be totally finite measures on $X$ such that $\nu << \mu$. Let $\lambda = \mu + \nu$ and $f = \frac{d\nu}{d\lambda}$. Show that $0 \leq f \leq 1$ $\lambda$-a.e. and that $0 \leq f < 1$ $\mu$-a.e.

**Solution.** Since $\mu$ and $\nu$ are totally finite, $f$ is integrable with respect to all measures. We have

$$\nu(E) = \int_E f \, d\lambda = \int_E f \, d\nu + \int_E f \, d\mu.$$

Let $E = \{f < 0\} \in S$. Then $0 \leq \nu(E) = \int_E f \, d\lambda \leq 0$. Theorem 5.3.7 implies that $\lambda(E) = 0$.

Let $A = \{f \geq 1\} \in S$. Then

$$\nu(A) = \int_A f \, d\nu + \int_A f \, d\mu \geq \int_A 1 \, d\nu + \int_A 1 \, d\mu = \nu(A) + \mu(A) \geq \nu(A).$$

Since $\nu$ is finite, it shows that $\mu(A) = 0$. Thus $0 \leq f < 1$ $\mu$-a.e.

Let $B = \{f > 1\} \in S$. Then $A \subset B$, so $\mu(B) = 0$. Thus

$$0 = \nu(B) - \nu(B) = \int_B f \, d\mu + \int_B f \, d\nu - \int_B 1 \, d\nu = \int_B (f - 1) \, d\nu.$$

Again, Theorem 5.3.7 implies that $\nu(B) = 0$. Therefore $\lambda(B) = 0$, which means $0 \leq f \leq 1$ $\lambda$-a.e. □

**Problem 3.** Let $X = (X, S)$ be a measurable space, $\mu$ and $\nu$ be totally $\sigma$-finite measures on $X$ such that $\nu << \mu$ and $\mu << \nu$. Let $\lambda = \mu + \nu$. Show that

$$\frac{d\nu}{d\mu} \frac{d\mu}{d\nu} = 1 \quad \lambda\text{- a.e.}$$

**Solution.** Denote $f = \frac{d\nu}{d\mu}$, $g = \frac{d\mu}{d\nu}$. First we assume that $\mu$ and $\nu$ are totally finite measures on $X$. Then by Corollary 6.5.2 for every $E \in S$ we have

$$\nu(E) = \int_E f \, d\mu = \int_E fg \, d\nu.$$

Thus, for every $E \in S$

$$0 = \nu(E) - \nu(E) = \int_E (1 - fg) \, d\nu.$$

Since measures are finite, all functions under consideration are integrable. Thus, by Lemma 5.3.8, $fg = 1$ $\nu$-a.e. Similarly, $fg = 1$ $\mu$-a.e. It implies $fg = 1$ $\lambda$-a.e.

Now, in general, there exist two sequences of measurable sets $\{E_i\}_{i\geq 1}$ and $\{F_i\}_{i\geq 1}$ such that

$$X = \bigcup_{i\geq 1} E_i = \bigcup_{i\geq 1} F_i \quad \text{and for every } i \geq 1 \quad \mu(E_i) < \infty, \ \nu(F_i) < \infty.$$

Thus

$$X = \bigcup_{i,j\geq 1} (E_i \cap F_j) \quad \text{and for every } i, j \geq 1 \quad \mu(E_i \cap F_j) < \infty, \quad \nu(E_i \cap F_j) < \infty.$$

Applying the first part separately for each $E_i \cap F_j$ we obtain the result. $\square$