Solutions of Assignment # 5.

Problem 1. Let λ denote the Lebesgue measure on $\mathbb{R}$. Is it true that for every sequence of bounded integrable functions $\{f_n\}_{n\geq 1}$ from A to $\mathbb{R}$ one has

$$\int_A \liminf_{n\to\infty} f_n \, d\lambda \leq \liminf_{n\to\infty} \int_A f_n \, d\lambda \, ,$$

where

a. $A = [0,1]$?

b. $A = \mathbb{R}$ and the sequence is uniformly bounded?

Solution.

a. No. Consider functions f_n defined by

$$f_n(x) = \begin{cases} -n, & x \in (0, 1/n) \\ 0, & \text{otherwise} . \end{cases}$$

Clearly, f_n is bounded, integrable, and $\int_A f_n \, d\lambda = -1$ for every n. On the other hand for $f_n(x)$ tends to 0 for every $x \in [0,1]$. Thus

$$0 = \int_A \liminf_{n\to\infty} f_n \, d\lambda > \liminf_{n\to\infty} \int_A f_n \, d\lambda = -1.$$

b. No. Consider functions f_n defined by

$$f_n(x) = \begin{cases} -1/n, & x \in (n, 2n) \\ 0, & \text{otherwise} . \end{cases}$$

Clearly, $|f_n|$ is bounded by 1 (so $\{f_n\}_n$ is uniformly bounded), integrable, and $\int_A f_n \, d\lambda = -1$ for every n. On the other hand for $f_n(x)$ tends to 0 for every $x \in \mathbb{R}$. Thus

$$0 = \int_A \liminf_{n\to\infty} f_n \, d\lambda > \liminf_{n\to\infty} \int_A f_n \, d\lambda = -1.$$

□

Problem 2. Let (X, S, μ) be a measure space, $E \in S$, $\mu(E) < \infty$. Let $f : E \to \bar{\mathbb{R}}$ be a measurable function satisfying $\mu(\{x \in E \mid |f(x)| > t\}) \leq 1/t^2$ for every $t > 1$. Is f integrable on E? Is the condition "$\mu(E) < \infty$" important?

Solution.

a. Yes, f is integrable on E. Indeed, by the distribution formula we have

$$\int_E |f| \, d\mu = \int_0^\infty \mu\{|f| > t\} \, dt = \int_0^2 \mu\{|f| > t\} \, dt + \int_2^\infty \mu\{|f| > t\} \, dt$$

$$\leq 2\mu\{|f| > 0\} + \int_2^\infty t^{-2} \, dt \leq 2\mu(E) + 1/2 < \infty.$$

b. Yes, the condition is important. Without this condition nothing can be said about integrability of f. Indeed, f can be integrable (just take $f(x) = 0$ for every $x \in E$), but can be non integrable as the following example shows. Let $f(x) = 1$ for every $x \in E$. Then for every $t > 1$,

$$\mu(\{x \in E \mid |f(x)| > t\}) = \mu(\emptyset) = 0,$$

but

$$\int_E |f| \, d\mu = \mu(E) = \infty.$$

□

Problem 3. Let (X, S, μ) be a measure space. Let $f : X \to \bar{\mathbb{R}}$ be a measurable function such that

$$\nu(E) = \int_E f \, d\mu$$

is defined for every $E \in S$. Describe a Hahn decomposition with respect to ν.

Solution. Consider $A = \{f \geq 0\}$ and $B = \{f < 0\}$. Clearly, $X = A \cup B$ and $A \cap B = \emptyset$. Since f is measurable, $B \in S$.

Let $E \in S$. Then $E \cap B \in S$ and $\nu(E) = \int_E f \, d\mu \leq 0$. It shows that B is negative.

Let $E \in S$. Then $E \cap A = E \setminus B \in S$ and $\nu(E) = \int_E f \, d\mu \geq 0$. It shows that A is positive.

Thus $X = A \cup B$ is a Hahn decomposition with respect to ν. □