Solutions of Assignment # 4.

Problem 1. Let $X = (\mathbb{R}, \mu)$, where μ is the Lebesgue measure. Let $f : X \to \mathbb{R}$ be continuous a.e. (that is, $\mu(\{x \,|\, f \text{ is not continuous at } x\}) = 0$). Show that f is Lebesgue measurable.

Solution. Denote $A = \{x \,|\, f \text{ is not continuous at } x\}$ and, given $q \in \mathbb{R}$, $L_q = \{x \mid f(x) < q\}$. Clearly, if $x \in L_q \cap A^c$, then, by continuity, there exists $\delta > 0$ such that $(x - \delta, x + \delta) \in L_q$. In other words

$$L_q \cap A^c \subset \text{int} L_q$$

(the interior of L_q). Therefore,

$$L_q = (L_q \cap A^c) \cup (L_q \cap A) = (\text{int} L_q) \cup (L_q \cap A).$$

The first set (the interior of L_q) is open, and, hence, Borel. The second set is subset of a set of measure zero and, thus, is Lebesgue measurable (as the Lebesgue measure is complete). Thus L_q is Lebesgue measurable. Since q is arbitrary, we obtain that f is Lebesgue measurable. □

Remark. Another way to solve this problem is to notice that f is continuous on A^c, so the preimage of any open set is open in induced topology on A^c, which implies that the preimage of any open set is Lebesgue measurable.

Problem 2. Show that for every Lebesgue measurable function $f : \mathbb{R} \to \mathbb{R}$ there exists a Borel measurable function $g : \mathbb{R} \to \mathbb{R}$ such that $f = g$ a.e. with respect to the Lebesgue measure.

Solution. As usual we denote the class of all Borel sets by $\mathcal{B}$ and the class of all Lebesgue measurable sets by $\mathcal{L}$. By results about the completion of a measure we know that every $L \in \mathcal{L}$ can be presented as $L = B \cup N$, where $B \in \mathcal{B}$, $N \subset D$ for some $D \in \mathcal{B}$ with $\mu(D) = 0$.

For every $q \in \mathbb{Q}$ we denote $L_q = \{x \mid f(x) < q\}$. Since f is Lebesgue measurable we have $L_q \in \mathcal{L}$. Hence it can be presented as $L_q = B_q \cup N_q$, where $B_q \in \mathcal{B}$, $N_q \subset D_q$ for some $D_q \in \mathcal{B}$ with $\mu(D_q) = 0$. Denote

$$M = \bigcup_{q \in \mathbb{Q}} D_q.$$

Since $\mathbb{Q}$ is countable we observe that $M \in \mathcal{B}$ and $\mu(M) = 0$. We define a function g by

$$g(x) = \begin{cases} 0, & x \in M \\ f(x), & x \in M^c. \end{cases}$$

Clearly, $f = g$ a.e. We show that g is Borel measurable. Let $q \in \mathbb{Q}$, $q \leq 0$. Then

$$\{x \mid g(x) < q\} = \{x \mid f(x) < q\} \cap M^c = L_q \cap M^c = (B_q \cup N_q) \cap M^c = B_q \cap M^c$$

is a Borel set (since B_q and M are, and since $N_q \subset M$). If $q \in \mathbb{Q}$, $q > 0$ then

$$\{x \mid g(x) < q\} = \{x \mid f(x) < q\} \cup M = L_q \cup M = B_q \cup M$$

is a Borel set. By a theorem in class (and a remark after) it implies that g is a Borel measurable function. □

Remark. Another way to solve this problem is to define g as

$$g(x) = \inf \{q \in \mathbb{Q} \mid x \in B_q\}$$

if the infimum is finite and $g(x) = 0$ otherwise.

Problem 3. Let (X, S, μ) be a measure space such that $\mu(X) < \infty$. Let $\{f_n\}_n$ be a sequence of measurable functions which is convergent in measure to a measurable function f. Show that $\{f_n^2\}_n$ is convergent in measure to f^2.

Solution. Fix $\varepsilon > 0$.

Given $q \geq 0$ denote $L_q = \{x \mid |f(x)| > q\}$. Since f is measurable, $L_q \in S$. Clearly $L_q \subset L_r$ if $q \geq r$, thus, by continuity and finiteness of μ we have

$$0 = \mu(\emptyset) = \mu\left(\bigcap_{k\geq 1} L_k\right) = \lim_{k\to\infty} \mu(L_k).$$

Thus for every $\delta > 0$ there exists k_δ such that

$$\mu(L_{k_\delta}) \leq \delta.$$

Since f_n tends to f in measure, for every $\delta > 0$ there exists $N = N(\delta)$ such that for every $n \geq N$ one has

$$\mu\left(\{x \mid |f_n(x) - f(x)| > k_\delta\}\right) \leq \delta.$$

Since

$$\{x \mid |f_n(x) + f(x)| > 3k_\delta\} \subset \{x \mid |f(x)| > k_\delta\} \cup \{x \mid |f_n(x) - f(x)| > k_\delta\},$$

it implies for $n > N$

$$\mu\left(\{x \mid |f_n(x) + f(x)| > 3k_\delta\}\right) \leq 2\delta.$$

Now note that

$$\{x \mid |f_n^2(x) - f^2(x)| > \varepsilon\} = \{x \mid |f_n(x) - f(x)| \cdot |f_n(x) + f(x)| > \varepsilon\}$$

$$\subset \{x \mid |f_n(x) - f(x)| > \varepsilon/(3k_\delta)\} \cup \{x \mid |f_n(x) + f(x)| > 3k_\delta\}.$$

Since f_n tends to f in measure, there exists $M = M(\varepsilon, \delta)$ such that for every $n > M$ one has

$$\mu\left(\{x \mid |f_n(x) - f(x)| > \varepsilon/(3k_\delta)\}\right) \leq \delta.$$

Choosing $N_0 = \max\{N, M\}$ we observe that for $n > M$

$$\mu\left(\{x \mid |f_n^2(x) - f^2(x)| > \varepsilon\}\right) \leq 3\delta.$$

It shows that

$$\lim_{n\to\infty} \mu\left(\{x \mid |f_n^2(x) - f^2(x)| > \varepsilon\}\right) = 0,$$

i.e., that f_n^2 tends to f^2 in measure. □