Solutions of Assignment # 3.

Problem 1. Let μ be a measure on a σ-ring S. Let $\bar{\mu}$ be its completion on $S_0 = \{A \cup B \,|\, A \in S, B \subset D \in S \text{ with } \mu(D) = 0\}$. Let $A, B \in S$ and E be such that $A \subset E \subset B$ and $\mu(B \setminus A) = 0$. Show that $E \in S_0$.

Solution. Note $E = A \cup (E \setminus A)$. Clearly, $E \setminus A \subset B \setminus A$. Since $A \in S$ and $\mu(B \setminus A) = 0$, we obtain $E \in S_0$. $\square$

Problem 2. Let μ be the Lebesgue measure on $\mathbb{R}$ and $E \subset \mathbb{R}$ be a Lebesgue measurable set. Show that there exist F_σ set A (that is, A can be presented as a countable union of closed sets) and G_δ set B (that is, B can be presented as a countable intersection of open sets) such that $A \subset E \subset B$ and $\mu(B \setminus A) = 0$. (Note that both sets, A and B, are Borel sets.)

Solution. Consider sets E_n, $n \geq 1$, defined by $E_n = E \cap A_n$, where $A_n = [n-1, n) \cup (-n, -n+1]$. Clearly, A_n is a Borel set for every $n \geq 1$ and $E = \cup_{n\geq 1} E_n$. Since any Borel set is Lebesgue measurable and E is Lebesgue measurable, we observe that E_n's are Lebesgue measurable. By a theorem in the class (Theorem 3.4.3), for every $n \geq 1$ and $k \geq 1$ there exists an open set B_{nk} such that $B_{nk} \supset E_k$ and

$$\mu(E_k) = \mu^*(E_k) \geq \mu(B_{nk}) - \frac{1}{n2^k}.$$

Since E_k, $k \geq 1$, are obviously of finite measure (actually $\mu(E_k) \leq \mu(A_k) = 2$), we obtain

$$\mu(B_{nk} \setminus E_k) = \mu(B_{nk}) - \mu(E_k) \leq \frac{1}{n2^k}.$$

Now for every $n \geq 1$ consider $B_n = \cup_{k\geq 1} B_{nk}$. Since every B_{nk} is open, B_n is open for every n. Since $B_{nk} \supset E_k$ for every k, n and $E = \cup_{k\geq 1} E_k$, we observe $B_n \supset E$. Since,

$$B_n \setminus E = \left(\bigcup_{k\geq 1} B_{nk} \right) \setminus E \subset \bigcup_{k\geq 1} (B_{nk} \setminus E_k)$$

we also have

$$\mu\left(B_n \setminus E\right) \leq \mu\left(\bigcup_{k\geq 1} (B_{nk} \setminus E_k) \right) \leq \sum_{k\geq 1} \mu\left(B_{nk} \setminus E_k\right) \leq \sum_{k\geq 1} \frac{1}{n2^k} = \frac{1}{n}.$$

Finally, take $B = \cap_{n\geq 1} B_n$. Then $B \supset E$ and for every n we have

$$\mu\left(B \setminus E\right) \leq \mu\left(B_n \setminus E\right) \leq 1/n.$$

It means $\mu\left(B \setminus E\right) = 0$.

Thus we proved that there are open sets B_n, $n \geq 1$, such that $B = \cap_{n\geq 1} B_n \supset E$ and $\mu\left(B \setminus E\right) = 0$.

Applying the same construction to the set E^c (note that E^c is Lebesgue measurable, since E is Lebesgue measurable), we obtain that there are open sets C_n, $n \geq 1$, such that $C = \cap_{n\geq 1} C_n \supset E^c$ and $\mu\left(C \setminus E^c\right) = 0$. Let $A_n = C_n^c$, $n \geq 1$. Then A_n, $n \geq 1$, are closed sets and

$$A = \bigcup_{n\geq 1} A_n = \left(\bigcap_{n\geq 1} C_n \right)^c = C^c \subset E.$$

Moreover, since $E \setminus A = A^c \setminus E^c = C \setminus E^c$, we have $\mu(E \setminus A) = 0$. It implies

$$\mu(B \setminus A) = \mu((B \setminus E) \cup (E \setminus A)) \leq \mu(B \setminus E) + \mu(E \setminus A) = 0,$$

which completes the proof. $\square$

Problem 3. Let μ be the Lebesgue measure on $\mathbb{R}$ and $E \subset \mathbb{R}$ be Lebesgue measurable set, which is not a Borel set. Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x, & x \in E \\ -x, & x \notin E. \end{cases}$$

Is f Borel measurable? Is f Lebesgue measurable?

Solution. First note that the function $g(x) = x$ is both Borel and Lebesgue measurable. Indeed, for every A we have $g^{-1}(A) = A$. Thus if A is a Borel set then $g^{-1}(A)$ is a Borel set. It shows that g is Borel measurable. Since any Borel set is Lebesgue measurable, any Borel measurable function is Lebesgue measurable as well.

a. Consider $h = \chi_E - \chi_{E^c}$. An exercise in the class says that the characteristic function of a set is measurable iff the set is measurable. Thus h is a difference of two Lebesgue measurable functions. Therefore, by a theorem in the class, h is Lebesgue measurable. By another theorem in the class the product of two measurable functions is measurable. Thus $f = gh$ is Lebesgue measurable.

b. Assume that f is a Borel measurable. Then, by a theorem in the class, the set $E = \{x \mid f(x) = g(x)\}$ is Borel measurable. Contradiction, which shows that f is not Borel measurable. $\square$

Answer. f is Lebesgue measurable, but not Borel measurable.