Solutions of Assignment # 2.

Problem 1. Let μ_1^*, μ_2^* be finite outer measures on $\mathcal{P}(X)$. Define μ^* on $\mathcal{P}(X)$ by $\mu^* = \mu_1^* + \mu_2^*$. As usual, let $\bar{S}$, $\bar{S}_1$, and $\bar{S}_2$ denote the classes of measurable sets of μ^*, μ_1^*, and μ_2^* correspondingly. Show that $\bar{S} = \bar{S}_1 \cap \bar{S}_2$.

Solution.

a. Let $A \in \bar{S}_1 \cap \bar{S}_2$. Then by definition of a measurable set we have for every $B \in \mathcal{P}(X)$

$$\mu^*(B) = \mu_1^*(B) + \mu_2^*(B) = \mu_1^*(B \cap A) + \mu_1^*(B \cap A^c) + \mu_2^*(B \cap A) + \mu_2^*(B \cap A^c)$$
$$= \mu^*(B \cap A) + \mu^*(B \cap A^c),$$

which means that $A \in \bar{S}$. Thus $\bar{S}_1 \cap \bar{S}_2 \subset \bar{S}$.

b. Let $A \in \bar{S}$. Using that the measures are finite and the definition of a μ^*-measurable set we have for every $B \in \mathcal{P}(X)$

$$\mu_1^*(B) = \mu^*(B) - \mu_2^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c) - \mu_2^*(B).$$

Since $B = ((B \cap A) \cup (B \cap A^c))$, using subadditivity of μ_2, we observe

$$\mu_1^*(B) \geq \mu^*(B \cap A) + \mu^*(B \cap A^c) - \mu_2^*(B \cap A) - \mu_2^*(B \cap A^c) = \mu_1^*(B \cap A) + \mu_1^*(B \cap A^c).$$

It shows that $A \in \bar{S}_1$. Similarly we obtain that $A \in \bar{S}_2$. It proves $\bar{S} \subset \bar{S}_1 \cap \bar{S}_2$ and therefore concludes the proof. $\square$

Remark. Another way to prove $\bar{S} \subset \bar{S}_1 \cap \bar{S}_2$: Let $A \in \bar{S}$ and $B \in \mathcal{P}(X)$. Then

$$\mu^*(B) = \mu^*(B \cap A) + \mu^*(B \cap A^c),$$

which means (since $\mu^* = \mu_1^* + \mu_2^*$)

$$\mu_1^*(B) + \mu_2^*(B) = \mu_1^*(B \cap A) + \mu_2^*(B \cap A) + \mu_1^*(B \cap A^c) + \mu_2^*(B \cap A^c)$$
$$= (\mu_1^*(B \cap A) + \mu_1^*(B \cap A^c)) + (\mu_2^*(B \cap A) + \mu_2^*(B \cap A^c)).$$

By subadditivity we have

$$\mu_1^*(B) \leq \mu_1^*(B \cap A) + \mu_1^*(B \cap A^c) \quad \text{and} \quad \mu_2^*(B) \leq \mu_2^*(B \cap A) + \mu_2^*(B \cap A^c).$$

Since all measures are finite, it implies

$$\mu_1^*(B) = \mu_1^*(B \cap A) + \mu_1^*(B \cap A^c) \quad \text{and} \quad \mu_2^*(B) = \mu_2^*(B \cap A) + \mu_2^*(B \cap A^c),$$

which means $A \in \bar{S}_1 \cap \bar{S}_2$.

Problem 2. Let μ^* be an outer measure on a hereditary σ-ring H. Let E be a μ^*-measurable set and $F \in H$. Show that

$$\mu^*(E) + \mu^*(F) = \mu^*(E \cap F) + \mu^*(E \cup F).$$

Solution. Since E is measurable we have

$$\mu^*(F) = \mu^*(F \cap E) + \mu^*(F \cap E^c)$$

and

$$\mu^*(F \cup E) = \mu^*\left((F \cup E) \cap E\right) + \mu^*\left((F \cup E) \cap E^c\right) = \mu^*(E) + \mu^*(F \cap E^c).$$

Thus

$$\mu^*(E \cup F) + \mu^*(E \cap F) = \mu^*(E) + \mu^*(F \cap E^c) + \mu^*(E \cap F) = \mu^*(E) + \mu^*(F).$$

□

Problem 3. Let $X = \mathbb{N}$ (the set of all positive integers). Define μ^* on $\mathcal{P}(X)$ by

$$\mu^*(E) = \limsup_{n\to\infty} \left(\frac{1}{n} \operatorname{card}\left(E \cap \{1, \ldots, n\}\right) \right).$$

Is μ^* an outer measure?

Solution. We show that μ^* is not countably subadditive, so it is not an outer measure. Let $A_k = \{k\}$ for every k. Clearly, $X = \cup_{k\geq 1} A_k$. Now, since for every n and k one has $X \cap \{1, \ldots, n\} = \{1, \ldots, n\}$ and $A_k \cap \{1, \ldots, n\} \subset \{k\}$, we observe

$$\mu^*(X) = \limsup_{n\to\infty} \left(\frac{1}{n}\, n \right) = 1 \quad \text{and} \quad 0 \leq \mu^*(A_k) \leq \limsup_{n\to\infty} \left(\frac{1}{n} \right) = 0.$$

Thus

$$1 = \mu^*(X) > \sum_{k=1}^{\infty} \mu^*(A_k) = \sum_{k=1}^{\infty} 0 = 0.$$

It proves our claim. □

Answer. No.