Solutions of Assignment # 1.

Problem 1. Let X be a set, $R \subset \mathcal{P}(X)$ be a σ-ring, μ be a measure on R. Show that the set

$$\{A \,|\, A \text{ is } \sigma\text{-finite}\}$$

is a σ-ring.

Solution. First note that $\emptyset \in A$, as $\emptyset \in R$ and $\mu(\emptyset) = 0$ (clearly, any set of finite measure is σ-finite).

Now, let $E, F \in A$. Then, by definition, $E, F \in R$, so $E \setminus F \in R$. Also, by definition, there are sets $E_1, E_2, \cdots \in R$ such that $E \subset \cup_{i \geq 1} E_i$ and $\mu(E_i) < \infty$ for every i. Clearly,

$$E \setminus F \subset E \subset \bigcup_{i \geq 1} E_i.$$

It shows $E \setminus F \in R$.

Finally, let $E_1, E_2, \cdots \in A$. Then, by definition, $E_i \in R$ for every i, hence $\cup_{i \geq 1} E_i \in R$. Also, by definition, there are sets $E_{ij} \in R$ such that for every $i \geq 1$

$$E_i \subset \bigcup_{j \geq 1} E_{ij}$$

and for every $i, j \geq 1$ the measure of E_{ij} is finite. Thus we have countable family of sets $\{E_{ij}\}$ of finite measure such that their union covers $\cup_i E_i$. It concludes the proof. □

Problem 2. Let X be a set, $R \subset \mathcal{P}(X)$ be a ring, μ be a measure on R. Introduce a relation "$\sim$" on R by $E \sim F$ iff $\mu(E \Delta F) = 0$.

a. Show that $E \sim F$ implies $\mu(E) = \mu(F) = \mu(E \cap F)$.

b. Is $\{E \,|\, E \sim \emptyset\}$ a ring?

Solution.

a. Let $E \sim F$. Since μ is a measure, $\mu(E \setminus F) \leq \mu(E \Delta F) = 0$ and $\mu(F \setminus E) \leq \mu(F \Delta E) = 0$. Thus $\mu(E \setminus F) = \mu(F \setminus E) = 0$. Therefore

$$\mu(E) = \mu((E \setminus F) \cup (E \cap F)) = \mu(E \setminus F) + \mu(E \cap F) = \mu(E \cap F)$$

Similarly, $\mu(F) = \mu(E \cap F)$.

b. We show that $R = \{E \,|\, E \sim \emptyset\}$ is a ring. First note that $E \Delta \emptyset = E$, thus $E \in R$ if and only if $\mu(E) = 0$. Clearly, $\emptyset \in R$, so $R \neq \emptyset$. Now, let $E, F \in R$. Then $\mu(E \setminus F) \leq \mu(E) = 0$ and $\mu(E \cup F) \leq \mu(E) + \mu(F) = 0$. Therefore, $E \setminus F \in R$ and $E \cup F \in R$. Thus R is a ring. □

Problem 3. Let X be a set, $R \subset \mathcal{P}(X)$ be a ring, μ be a measure on R. Let $\{E_i\}_{i=1}^{\infty}$ be a sequence in R such that

$$\liminf_{i \to \infty} E_i \in R \quad \text{and for every } n \text{ one has } \bigcap_{i=n}^{\infty} E_i \in R.$$

Show that

$$\mu\left(\liminf_{i \to \infty} E_i\right) \leq \liminf_{i \to \infty} \mu(E_i).$$

Solution. Note that for every $n \geq 1$ and every $k \geq n$ we have

$$\bigcap_{i \geq n} E_i \subset E_k.$$

Since μ is a measure, it implies for every $k \geq n \geq 1$

$$\mu\left(\bigcap_{i \geq n} E_i\right) \leq \mu\left(E_k\right).$$

Taking infimum over $k \geq n$ we observe

$$\mu\left(\bigcap_{i \geq n} E_i\right) \leq \inf_{k \geq n} \mu\left(E_k\right),$$

for every $n \geq 1$.

Now observe that

$$\left\{\bigcap_{i \geq n} E_i\right\}_{n=1}^{\infty},$$

is an increasing sequence. By continuity of a measure, we obtain

$$\mu\left(\liminf_{n\to\infty} E_n\right) = \mu\left(\bigcup_{n \geq 1} \bigcap_{i \geq n} E_i\right) = \mu\left(\lim_{n\to\infty} \bigcap_{i \geq n} E_i\right)$$

$$= \lim_{n\to\infty} \mu\left(\bigcap_{i \geq n} E_i\right) \leq \lim_{n\to\infty} \inf_{k \geq n} \mu\left(E_k\right) = \liminf_{i\to\infty} \mu(E_i).$$

$\square$