Assignment # 3.

Due Feb. 23

Problem 1. Let μ be a measure on a σ-ring S. Let $\bar{\mu}$ be its completion on $S_0 = \{A \cup B \,|\, A \in S, B \subset D \in S \text{ with } \mu(D) = 0\}$. Let $A, B \in S$ and E be such that $A \subset E \subset B$ and $\mu(B \setminus A) = 0$. Show that $E \in S_0$.

Problem 2. Let μ be the Lebesgue measure on $\mathbb{R}$ and $E \subset \mathbb{R}$ be a Lebesgue measurable set. Show that there exist F_σ set A (that is, A can be presented as a countable union of closed sets) and G_δ set B (that is, B can be presented as a countable intersection of open sets) such that $A \subset E \subset B$ and $\mu(B \setminus A) = 0$. (Note that both sets, A and B, are Borel sets.)

Problem 3. Let μ be the Lebesgue measure on $\mathbb{R}$ and $E \subset \mathbb{R}$ be Lebesgue measurable set, which is not a Borel set. Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x, & x \in E \\ -x, & x \notin E. \end{cases}$$

Is f Borel measurable? Is f Lebesgue measurable?