Solutions of Midterm problems.

Problem 3. Is the condition "measure is finite" in the Egoroff Theorem necessary?

Solution. We show that the answer is negative. We consider $\mathbb{R}$ with the lebesgue measure λ. Let f_n be the characteristic function of the segment $[n, n+1]$ and f be identically equal to 0. Then, clearly, for every (fixed) x we have $f_n(x) \to f(x) = 0$. Now we show that $\{f_n\}$ does not converge almost uniformly. Assume that $\{f_n\}$ converges almost uniformly to a function g. Since almost uniform convergence implies pointwise a.e. convergence, we obtain $f = g$ a.e., which in turns implies that $\{f_n\}$ converges almost uniformly to f. By the definition of almost uniform convergence for $\varepsilon = 1/2$ there exists a set F of measure at most $1/2$ such that $\{f_n\}$ converges uniformly to f on F^c. But for $\delta = 1$ and every N we can find $n \geq N$ (say $n = N$) and $x \in E := [n, n+1] \setminus F$ (note $\lambda(E) \geq 1 - \lambda(F) \geq 1/2$) such that $|f_n(x) - f(x)| = 1 \geq \delta$. It contradicts to the uniform convergence on F^c. □

Problem 4. Let λ be the Lebesgue measure on $\mathbb{R}$. Show that for every $\varepsilon > 0$ and every Lebesgue measurable set E there exists an open F such that $E \subset F$ and $\lambda(F \setminus E) < \varepsilon$ (you may use that $\lambda^*(A) = \inf\{\lambda(B) \ : \ A \subset B, \ B \text{ is open}\}$).

Solution. For every n consider $E_n = E \cap (-n, n)$. By the formula given in Problem 4 there exists an open set $B_n \supset E_n$ such that $\lambda(E_n) + \varepsilon/2^n \geq \lambda(B_n)$ (note that we deal with measurable sets, so $\lambda = \lambda^*$). Since E_n is of finite measure (less than $2n$), we obtain

$$\lambda(B_n \setminus E_n) = \lambda(B_n) - \lambda(E_n) < \varepsilon/2^n.$$

Finally consider $F = \cup_n B_n$, which is open as union of open sets. Clearly,

$$E \subset F \quad \text{and} \quad F \setminus E \subset \bigcup_{n \geq 1} (B_n \setminus E_n).$$

Therefore

$$\lambda(F \setminus E) \leq \sum_{n \geq 1} \lambda(B_n \setminus E_n) < \sum_{n \geq 1} \varepsilon/2^n = \varepsilon,$$

which proves the statement. □

Problem 5. Let f be a function. Is it true that
a. f is measurable if and only if f^+ and f^- are measurable.
b. f is measurable if and only if $|f|$ is measurable.

Solution.
a. We show that answer is yes. First assume that f is measurable. Then $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$ are measurable as maxima of two measurable functions by an exercise in the class (note $g = 0$ is measurable as a constant function, by another exercise). On the other hand, if f^+ and f^- are measurable then $f = f^+ - f^-$ is measurable as difference of two measurable functions.
b. We show that answer is no. Let B be a non Lebesgue measurable set in $\mathbb{R}$ (exists by a statement

in the class). Let f be defined by $f(x) = 1$ for $x \in B$ and $f(x) = -1$ otherwise. Then $\{f > 0\} = B$, which is not Lebesgue measurable, so f is not measurable. On the other hand, $|f| = 1$ at every point, so it is measurable. $\square$

Problem 6. Does convergence in measure imply pointwise a.e. convergence?

Solution. No. Consider the following example. For every $n \geq 1$ and every $1 \leq k \leq n$ define the function g_{nk} as the characteristic function of the segment $[(k-1)/n, k/n]$ (we can consider functions on $\mathbb{R}$ or just on $[0,1]$ with Lebesgue measure). Let $\{f_n\}$ be the sequence $g_{11}, g_{21}, g_{22}, g_{31}, g_{32}, g_{33}, g_{41}, \ldots$. Then for every $\varepsilon > 0$ and every $n > 1/\varepsilon$ one has

$$\mu\left(\{|g_{nk}| > \varepsilon\}\right) = \mu\left(\{[(k-1)/n, k/n]\}\right) = 1/n \to 0,$$

which shows that $\{f_n\}$ converges in measure to 0. However for EVERY fixed $x \in [0,1]$ the sequence $\{f_n(x)\}$ is not convergent, since it has infinitely many zeros and ones. Thus, $\{f_n\}$ is not pointwisely convergent. $\square$

Problem 7. Let μ be a finite measure on Borel subsets of $[0,1]$ satisfying $\mu(\{x\}) = 0$ for every x. Show that for every $\varepsilon > 0$ there exists a dense open F such that $\mu(F) < \varepsilon$.

Solution. First we show that for every $a \in (0,1)$ and every δ there exists b (which is smaller than a and $1-a$) such that $(a-b, a+b)$ has measure smaller than δ. Indeed, for all large enough n (so that $n > 1/a$ and $n > 1/(1-a)$) consider $b_n = 1/n$ and $I_n = (a - b_n, a + b_n)$. Then $I_n \supset I_{n+1}$ for every n and $\{a\} = \cap_n I_n$. Since μ is finite, we obtain by continuity,

$$0 = \mu\left(a\right) = \mu\left(\cap_n I_n\right) = \lim_{n\to\infty} \mu\left(I_n\right).$$

It shows that for every δ there exists n with $\mu\left(I_n\right) < \delta$.

Now consider $A = \mathbb{Q} \cap (0,1)$. It is countable and dense in $[0,1]$. Since it is countable we can write it as a sequence $\{a_k\}_k$. Fix $\varepsilon > 0$. By above, for every k there exists b_k such that $J_k = (a_k - b_k, a_k + b_k) \subset (0,1)$ and has measure smaller than $\varepsilon/2^k$. Define F as union of J_k's. Since every J_k is open, F is open. Since $J \supset A$, J is dense in $[0,1]$. Finally

$$\mu(F) \leq \sum_{k\geq 1} \mu(J_k) < \sum_{k\geq 1} \varepsilon/2^k = \varepsilon,$$

which proves the desired result. $\square$