Solutions of Midterm problems.

Problem 6. Let $\mathcal{F}$ be a set of Lebesgue measurable functions. Let F be the function, defined by $F(x) = \sup\{f(x) \mid f \in \mathcal{F}\}$. Is F necessarily Lebesgue measurable?

Solution. No. Consider the following example. Let A be a non Lebesgue measurable set (exists, as we discussed in class). Given $z \in A$, let f_z denote the characteristic function of $\{z\}$ (that is $f_z(z) = 1$ and $f_z(x) = 0$ for every $x \neq z$). Since every singleton is a Borel set, f_z is Borel (and, hence, Lebesgue) measurable for every z. Now consider

$$\mathcal{F} = \{f_z\}_{z \in A}.$$

Clearly, $F(x) = \sup\{f(x) \mid f \in \mathcal{F}\}$ is the characteristic function of A. Since A is not measurable, $F(x)$ is also not measurable (by an exercise in the class, or just notice that $\{x \mid F(x) \geq 1\} = A$ is a non measurable set.) $\square$

Problem 7. Let $\{f_n\}_n$ be a sequence of measurable functions, convergent in measure to measurable functions f and g. Show that $f = g$ a.e.

Solution. For every $\varepsilon > 0$ we have

$$\{x \mid |f(x) - g(x)| > \varepsilon\} \subset \{x \mid |f(x) - f_n(x)| > \varepsilon/2\} \cup \{x \mid |f_n(x) - g(x)| > \varepsilon/2\}.$$

Since f_n tends to f and to g in measure we observe

$$\mu\left(\{x \mid |f(x) - g(x)| > \varepsilon\}\right)$$

$$\leq \mu\left(\{x \mid |f(x) - f_n(x)| > \varepsilon/2\}\right) + \mu\left(\{x \mid |f_n(x) - g(x)| > \varepsilon/2\}\right) \to 0 \quad \text{as} \quad n \to \infty.$$

It shows that for every $\varepsilon > 0$ one has $\mu\left(\{x \mid |f(x) - g(x)| > \varepsilon\}\right) = 0$. Since

$$\{x \mid f(x) \neq g(x)\} = \bigcup_{k \geq 1} \{x \mid |f(x) - g(x)| > 1/k\},$$

the result follows. $\square$

Problem 8. Let $X = (X, S, \mu)$ be a measure space. Let $\{E_n\}_n$ be a sequence of measurable sets such that $\mu(\cup_{n \geq m} E_n) < \infty$ for some m. Show that

$$\mu\left(\limsup_{n \to \infty} E_n\right) \geq \limsup_{n \to \infty} \mu(E_n).$$

Is the condition "$\mu(\cup_{n \geq m} E_n) < \infty$ for some m" needed?

Solution. A proof of the inequality repeats the solution of Problem 3 in Assignment 1 with obvious modifications. We provide an example showing that one can't omit the condition "$\mu(\cup_{n \geq m} E_n) < \infty$". Consider $X = \mathbb{N}$ with the counting measure. Let $E_n = \{n\}$. Then $\mu(E_n) = 1$ for every n, and $\cup_{m \geq n} E_n = \{n, n+1, n+2, \ldots\}$. Thus

$$\limsup_{n\to\infty} E_n = \bigcap_{n\geq 1}\bigcup_{m\geq n} E_n = \emptyset,$$

which implies

$$0 = \mu\left(\limsup_{n\to\infty} E_n\right) < \limsup_{n\to\infty} \mu(E_n) = 1.$$

Similar examples can be constructed for the Lebesgue measure on $\mathbb{R}$, say take $E_n = (n, n+1]$ or $E_n = (n, 2n]$. □

Problem 9. Let μ be a finite measure of X. Let $\{f_n\}_n$ be a sequence of finite measurable functions, convergent in measure to a finite measurable function f. Show that for every finite measurable function g the sequence $\{gf_n\}_n$ converges in measure to gf. Is the condition "μ is finite" needed?

Solution. First we show that the condition "μ is finite" is needed. Consider $\mathbb{R}$ with the Lebesgue measure. Consider functions f_n, f, g defined by $f_n(x) = 1/n$, $f(x) = 0$, $g(x) = x$ for every x. Then, for a fixed $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu\left(\{x \mid |f(x) - f_n(x)| > \varepsilon\}\right) = 0$$

(in fact, $\{x \mid |f(x) - f_n(x)| > \varepsilon\} = \emptyset$ for $n > 1/\varepsilon$). It shows that f_n tends to f in measure. On the other hand for $\varepsilon = 1$ (and, in fact, for every $\varepsilon > 0$)

$$\mu\left(\{x \mid |f(x)g(x) - g(x)f_n(x)| > \varepsilon\}\right) = \mu\left(\{x \mid |x/n| > \varepsilon\}\right) = \mu\left((-\infty, -n\varepsilon) \cup (n\varepsilon, \infty)\right) = \infty,$$

which means that f_ng does not tend to fg in measure.

Now we show the result for finite measures. Fix $\varepsilon > 0$.

Given $q \geq 0$ denote $L_q = \{x \mid |g(x)| > q\}$. Since g is measurable, L_q is measurable. Clearly $L_q \subset L_r$ if $q \geq r$, thus, by continuity and finiteness of μ we have

$$0 = \mu(\emptyset) = \mu\left(\bigcap_{k\geq 1} L_k\right) = \lim_{k\to\infty} \mu(L_k).$$

Thus for every $\delta > 0$ there exists k_δ such that $\mu(L_{k_\delta}) \leq \delta$.

Now note that

$$\{x \mid |g(x)f_n(x) - g(x)f(x)| > \varepsilon\} = \{x \mid |f_n(x) - f(x)| \cdot |g(x)| > \varepsilon\}$$

$$\subset \{x \mid |f_n(x) - f(x)| > \varepsilon/k_\delta\} \cup \{x \mid |g(x)| > k_\delta\}.$$

Since f_n tends to f in measure, there exists $N = N(\varepsilon, \delta)$ such that for every $n > N$ one has $\mu\left(\{x \mid |f_n(x) - f(x)| > \varepsilon/k_\delta\}\right) \leq \delta$. It implies that for $n > N$

$$\mu\left(\{x \mid |g(x)f_n(x) - g(x)f(x)| > \varepsilon\}\right) \leq 2\delta.$$

Thus $\lim_{n\to\infty} \mu\left(\{x \mid |g(x)f_n(x) - g(x)f(x)| > \varepsilon\}\right) = 0$. □