Quiz # 4.

Problem 2. Let f, g be functions on $\mathbb{R}$ defined by $f(t) = \sin t$ for every t; $g(t) = t$ for $t \in [0, 2\pi]$, $g(t) = 0$ for $t \notin [0, 2\pi]$. Find the convolution of f and g.

Solution.
Way 1.

$$f * g(x) = \int_{\mathbb{R}} f(y)g(x-y)\ dy.$$

Since $g(x-y) = 0$, whenever $x - y \notin [0, 2\pi]$ (that is $y \notin [x-2\pi, x]$),

$$f * g(x) = \int_{x-2\pi}^{x} (x-y)\sin y\ dy = x\int_{x-2\pi}^{x} \sin y\ dy - \int_{x-2\pi}^{x} y \sin y\ dy.$$

Using

$$\int z\sin z\ dz = \begin{bmatrix} u = z & du = dz \\ dv = \sin z\ dz & v = -\cos z \end{bmatrix} = -z\cos z + \int \cos z\ dz = -z\cos z + \sin z,$$

we obtain

$$f * g(x) = -x\cos y\Big|_{y=x-2\pi}^{y=x} + y\cos y\Big|_{y=x-2\pi}^{y=x} - \sin y\Big|_{y=x-2\pi}^{y=x}$$

$$= -x\left(\cos x - \cos(x-2\pi)\right) + \left(x\cos x - (x-2\pi)\cos(x-2\pi)\right) - \left(\sin x - \sin(x-2\pi)\right) = 2\pi\cos x.$$

Way 2.

$$f * g(x) = g * f(x) = \int_{\mathbb{R}} g(y)f(x-y)\ dy = \int_0^{2\pi} y\sin(x-y)\ dy$$

$$= \begin{bmatrix} u = y & du = dy \\ dv = \sin(x-y)\ dy & v = \cos(x-y) \end{bmatrix} = y\cos(x-y)\Big|_{y=0}^{y=2\pi} - \int_0^{2\pi} \cos(x-y)\ dy$$

$$= 2\pi\cos(x-2\pi) + \sin(x-y)\Big|_{y=0}^{y=2\pi} = 2\pi\cos x + \sin(x-2\pi) - \sin x = 2\pi\cos x.$$

□

Answer. $f * g(x) = 2\pi\cos x$.

Problem 2. Let $H = L_2([-1,1], \lambda)$, where λ is the Lebesgue measure. Let f, g, h be functions on $[-1, 1]$ defined by $f(t) = 1$ for every t, $g(t) = \text{sign } t$, and $h(t) = t$. Find the orthogonal projection of h on the span of f and g.

Solution. First note that f and g are orthogonal. Indeed,

$$(f, g) = \int_{-1}^{1} f(t)\bar{g}(t)\ dt = \int_{-1}^{0} (-1)\ dt + \int_0^1 1\ dt = 0.$$

Thus they are linearly independent (both are non-zero in L_2) and therefore they form an orthogonal basis of $E := \text{span}\{f, g\}$ (clearly, span of 2 vectors is at most 2 dimensional, so any two linearly independent vectors form a basis). Recall the general formula of the orthogonal projection onto span of n orthogonal non-zero vectors in a Hilbert space:

$$Px = \sum_{k=1}^{n} \frac{(x, e_k)}{\|e_k\|^2}\ e_k.$$

Thus, in our case we have

$$P_E h = \frac{(h, f)}{\|f\|^2}\ f + \frac{(h, g)}{\|g\|^2}\ g.$$

Now we compute the corresponding inner products and norms:

$$(h, f) = \int_{-1}^{1} h(t)\bar{f}(t)\ dt = \int_{-1}^{1} t\ dt = \frac{t^2}{2}\Big|_{-1}^{1} = 0,$$

$$(h, g) = \int_{-1}^{1} h(t)\bar{g}(t)\ dt = \int_{-1}^{0} (-t)\ dt + \int_{0}^{1} t\ dt = -\frac{t^2}{2}\Big|_{-1}^{0} + \frac{t^2}{2}\Big|_{0}^{1} = 1,$$

$$\|g\|^2 = \int_{-1}^{1} |g(t)|^2\ dt = \int_{-1}^{1} 1\ dt = 2.$$

Hence,

$$P_E h = 0 + \frac{1}{2} g.$$

□

Answer. $Ph = g/2$ (that is, $(Ph)(t) = \frac{1}{2}\text{sign}\ t$ for $t \in [-1, 1]$).