Quiz # 3.

In solutions below definitions are omitted.

Problem 1. Is $\ell_6 \subset \ell_4$?

Solution. We show that the **ANSWER** is **NO**.

Consider $x = \{x_n\}_{n\geq 1}$ defined by $x_n = 1/n^{1/4}$. Then

$$\sum_{n=1}^{\infty} |x_n|^6 = \sum_{n=1}^{\infty} 1/n^{3/2} < \infty,$$

so $x \in \ell_6$, but

$$\sum_{n=1}^{\infty} |x_n|^4 = \sum_{n=1}^{\infty} 1/n = \infty,$$

so $x \notin \ell_4$

$\square$

Remark. In general, $\ell_q \not\subset \ell_p$ for $p < q$: Consider $x = \{x_n\}_{n\geq 1}$ with $x_n = 1/n^{1/p}$, then, as above, $x \notin \ell_p$ and, clearly, $x \in \ell_q$ (proving this don't forget about the case $q = \infty$).

Problem 2. Let (Ω, μ) be a measure space. Let f and g be two measurable functions. Assume p, r, q be positive real numbers such that $1/r = 1/p + 1/q$. Show that

$$\left(\int_\Omega |fg|^r d\mu\right)^{1/r} \leq \left(\int_\Omega |f|^p d\mu\right)^{1/p} \left(\int_\Omega |g|^q d\mu\right)^{1/q}.$$

Solution. We have $1 = r/p + r/q$ for positive real numbers p, q, r. Thus, if we define $s = p/r$ then $s > 1$ and $s' = q/r$. Therefore, applying Hölder inequality to functions $|f|^r$ and $|g|^r$ we obtain

$$\int_\Omega |f|^r |g|^r d\mu \leq \left(\int_\Omega |f|^{rs} d\mu\right)^{1/s} \left(\int_\Omega |g|^{rs'} d\mu\right)^{1/s'} = \left(\int_\Omega |f|^p d\mu\right)^{r/p} \left(\int_\Omega |g|^q d\mu\right)^{r/q},$$

which implies the result. $\square$