Quiz # 2.

In solutions below definitions are omitted.

Problem 1. Let $0 \le a < b$ and define f on $[a, b]$ by $f(x) = x^2$. Is f absolutely continuous?

Solution. We show that the **ANSWER** is **YES**.

Let $\varepsilon > 0$. Choose $\delta = \varepsilon/2b$. Assume that

$$\sum_{i \ge 1} (b_i - a_i) < \delta,$$

where $a \le a_i < b_i \le b$ for every i. Then

$$\sum_{i \ge 1} |f(b_i) - f(a_i)| = \sum_{i \ge 1} (b_i^2 - a_i^2) = \sum_{i \ge 1} (b_i - a_i)(b_i + a_i) \le 2b \sum_{i \ge 1} (b_i - a_i) < 2b\delta = \varepsilon.$$

Thus, by the definition, f is absolutely continuous. □

Problem 2. Give the definition of a convex function. Let f be a convex function on $\mathbb{R}$ and $x < y < z$. Show that

$$\frac{f(y) - f(x)}{y - x} \le \frac{f(z) - f(y)}{z - y}.$$

Solution. Since $y \in (x, z)$ there exists $\lambda \in (0, 1)$ such that $y = \lambda x + (1 - \lambda) z$ (in fact, $\lambda = (z - y)/(z - x)$, but we don't use that). Thus, $y - x = (1 - \lambda)(z - x)$, $z - y = \lambda(z - x)$, and, hence, we have to prove that

$$\frac{f(y) - f(x)}{(1 - \lambda)(z - x)} \le \frac{f(z) - f(y)}{\lambda(z - x)},$$

which is equivalent to

$$\lambda \left(f(y) - f(x) \right) \le (1 - \lambda) \left(f(z) - f(y) \right).$$

(note, $\lambda > 0$, $1 - \lambda > 0$, $y - x > 0$). The later is equivalent to

$$f(y) \le \lambda f(x) + (1 - \lambda) f(z),$$

which holds by the definition of a convex function (recall, $y = \lambda x + (1 - \lambda) z$). □