Quiz # 1.

In solutions below definitions are omitted.

Problem 1. Let $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Find $D_+ f(0)$.

Solution.
$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} h \sin(1/h) = 0,$$
since $-h \le h\sin(1/h) \le h$. Thus f' exists at 0, $f'(0) = 0$, and, hence,
$$D_+ f(0) = f'(0) = 0.$$

□

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by
$$f(x) = \begin{cases} 0, & x \in (-\infty, 0], \\ x, & x \in (0,1), \\ 2, & x \in [1, \infty). \end{cases}$$
Let λ_f be the corresponding Lebesgue-Stieltjes measure. Find $\int_{\mathbb{R}} x \; d\lambda_f(x)$.

Solution. First note that $\lambda_f((-\infty, 0])$. Indeed,
$$\lambda_f((-\infty,0]) = \lambda_f\left(\bigcup_{n=1}^{\infty} (-n, 0]\right) \le \sum_{n=1}^{\infty} \alpha_f((-n,0]) = 0.$$
Therefore, the function $g(x) = x \ge 0$ λ_f-a.e. on $\mathbb{R}$ and we can use the distribution formula
$$\int_{\mathbb{R}} x \; d\lambda_f(x) = \int_{[0,\infty)} g(x) \; d\lambda_f(x) = \int_{[0,\infty)} F_g(t) \; d\lambda(t)$$
$$= \int_0^{\infty} \lambda_f(\{x \mid g(x) = x > t\}) \; dt = \int_0^{\infty} \lambda_f((t,\infty)) \; dt.$$
Now, for every $t > 0$ we compute $\lambda_f((t,\infty))$.

If $t \ge 1$ then $\lambda_f((t,\infty)) = 0$. Indeed,
$$\lambda_f((t,\infty)) = \lambda_f\left(\bigcup_{n=1}^{\infty} (t, t+n]\right) \le \sum_{n=1}^{\infty} \alpha_f(t, t+n]) = 0.$$

If $t < 1$, then, since the function f is right-continuous, and since by above $\lambda_f((1,\infty)) = 0$, we have
$$\lambda_f((t,\infty)) = \lambda_f((t,1] \cup (1,\infty)) = \lambda_f((t,1]) + \lambda_f((1,\infty)) = \alpha_f((t,1]) + 0 = f(t) - f(t) = 2 - t.$$

It implies,
$$\int_{\mathbb{R}} x \; d\lambda_f(x) = \int_0^{\infty} \lambda_f(\{(t,\infty\}) \; dt = \int_0^1 (2-t) \; dt = 2 - 1/2 = 3/2.$$

□

Answer. $\int_{\mathbb{R}} x \; d\lambda_f(x) = 3/2$.

Remark. Note that λ_f coincides with the Lebesgue measure on $(0,1)$ but not on $(0,1]$. Note also that $\lambda_f(1) = 1$. Please review the notion and properties of Lebesgue-Stieltjes measure as well as the applications of the distribution formula. In particular, please note that the distribution formula holds for NONNEGATIVE functions! Note that $g(x) = x$ is NOT a nonnegative function on $\mathbb{R}$.