Additional exercises for Chapter 3.

Problem 1. Let X be a normed space, $x \in X$, and $\{x_i\}_{i=1}^{\infty}$ be a sequence in X such that

$$\lim_{i\to\infty} x_i = x.$$

Show that

$$\lim_{i\to\infty} \|x_i\| = \|x\|.$$

Problem 2. Let X be a normed space and let $\sum_{i=1}^{\infty} x_i$ be a convergent series in X. Show that

$$\left\| \sum_{i=1}^{\infty} x_i \right\| \leq \sum_{i=1}^{\infty} \|x_i\| .$$

Problem 3. Given $1 \leq p < \infty$ and $x = \{x_i\}_{i=1}^{n} \in \mathbb{K}^n$, denote

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} .$$

Show that for every p, q satisfying $1 \leq p < q < \infty$ and every $x \in \mathbb{K}^n$ one has

$$\|x\|_p \leq n^{1/p-1/q} \|x\|_q.$$

Problem 4. Let (Ω, μ) be a measure space. Assume p, r, q be positive real numbers such that $1/r = 1/p + 1/q$. Show that for every two measurable functions f and g one has

$$\left(\int_\Omega |fg|^r \right)^{1/r} \leq \left(\int_\Omega |f|^p \right)^{1/p} \left(\int_\Omega |g|^q \right)^{1/q}$$

Problem 5. Let $1 \leq p < q \leq \infty$. Show that $\ell_q \not\subset \ell_p$.

Problem 6. Let λ denote the Lebesgue measure on $\mathbb{R}$. Show that for every $p \neq q$ one has $L_p(\mathbb{R}, \lambda) \not\subset L_q(\mathbb{R}, \lambda)$.

Problem 7. Let X be the space of continuous functions $f : [0, 1] \to \mathbb{R}$ endowed with the norm

$$\|f\|_X = \int_0^1 |f(t)|\, dt.$$

Is X complete?

Problem 8. Let $p \in [1, \infty]$. Let $\Omega = \{\omega\}$ be a singleton. Define a measure μ on Ω by $\mu(\Omega) = \infty$. Describe $L_p(\Omega, \mu)$. Is it true that for every $f \in L_\infty(\Omega, \mu)$ one has

$$\|f\|_\infty = \sup \left| \int_\Omega fg \; d\mu \right| ,$$

where supremum is taken over all $g \in L_1(\Omega, \mu)$ with $\|g\|_1 \leq 1$?

Problem 9. Show that

$$\mu\left(\{|f| > \|f\|_\infty\}\right) = 0.$$