14. a. Show that the sum and difference of two absolutely continuous functions are also absolutely continuous.

b. Show that the product of two absolutely continuous functions is absolutely continuous. [Hint: Make use of the fact that they are bounded.]

c. If f is absolutely continuous on $[a, b]$ and if f is never zero there, then the function $g = 1/f$ is also absolutely continuous on $[a, b]$.

15. The Cantor ternary function (Problem 2.48) is continuous and monotone but not absolutely continuous.

16. A monotone function f on $[a, b]$ is called **singular** if $f' = 0$ a.e.

a. Show that any monotone increasing function is the sum of an absolutely continuous function and a singular function.

b. Let f be a nondecreasing singular function on $[a, b]$. Then f has the following property: (S) Given $\epsilon > 0$, $\delta > 0$, there is a finite collection $\{[y_k, x_k]\}$ of nonoverlapping intervals such that

$$\sum |x_k - y_k| < \delta$$

and

$$\sum f(x_k) - f(y_k) > f(b) - f(a) - \epsilon.$$

[Hint: See proof of Lemma 13.]

c. Let f be a nondecreasing function on $[a, b]$ with property (S) of part (b). Then f is singular. [Hint: Use part (a).]

d. Let $\langle f_n \rangle$ be a sequence of nondecreasing singular functions on $[a, b]$ such that the function

$$f(x) = \sum f_n(x)$$

is everywhere finite. Then f is also singular.

e. Show that there is a strictly increasing singular function on $[0, 1]$.

17. a. Let F be absolutely continuous on $[c, d]$ and g be absolutely continuous with $c \le g \le d$ on $[a, b]$. Then $F \circ g$ is absolutely continuous on $[a, b]$.

b. Let $E = \{x: g'(x) = 0\}$. Then $m(g[E]) = 0$.

18. Let g be an absolutely continuous monotone function on $[0, 1]$ and E a set of measure zero. Then $g[E]$ has measure zero.

19. a. Construct an absolutely continuous strictly monotone function g on $[0, 1]$ such that $g' = 0$ on a set of positive measure. [Hint: Let G be the complement of a generalized Cantor set of positive measure (Problem 3.14), and let g be the indefinite integral of χ_G.]

b. Show that there is a set E of measure zero such that $g^{-1}[E]$ is not measurable. How does this example compare with that of Problem 3.28?

20. A function f is said to satisfy a Lipschitz condition on an interval if there is a constant M such that $|f(x) - f(y)| \leq M|x - y|$ for all x and y in the interval.

a. Show that a function satisfying a Lipschitz condition is absolutely continuous.

b. Show that an absolutely continuous function f satisfies a Lipschitz condition if and only if $|f'|$ is bounded.

c. Show that f satisfies a Lipschitz condition if one of its derivates (say D^+) is bounded.

21. *Change of variable.* Let g be a monotone increasing absolutely continuous function on $[a, b]$ with $g(a) = c$, $g(b) = d$.

a. Show that for any open set $O \subset [c, d]$

$$mO = \int_{g^{-1}[O]} g'(x)\, dx.$$

b. Let $H = \{x: g'(x) \neq 0\}$. If E is a subset of $[c, d]$ with $mE = 0$, then $g^{-1}(E) \cap H$ has measure zero.

c. If E is a measurable subset of $[c, d]$, then $F = g^{-1}[E] \cap H$ is measurable and

$$mE = \int_F g' = \int_a^b \chi_E(g(x))g'(x)\, dx.$$

d. If f is a nonnegative measurable function on $[c, d]$, then $(f \circ g)\, g'$ is measurable on $[a, b]$ and

$$\int_c^d f(y)\, dy = \int_a^b f(g(x))g'(x)\, dx.$$

22. *Change of variable II.* Let g be a monotone increasing absolutely continuous function on $[a, b]$ with $g(a) = c$, $g(b) = d$, and let f be an integrable function on $[c, d]$. Let

$$F(y) = \int_c^y f(t)\, dt,$$

and set $H(x) = F(g(x))$.

a. Show that H is absolutely continuous (Problem 17a) and that $F'(g(x))$ exists wherever H' and g' exist and $g'(x) \neq 0$. Thus

$$H'(x) = F'(g(x))g'(x)$$

almost everywhere except on the set E where $g'(x) = 0$.

b. Let

$$f_0(y) = \begin{cases} f(y) & y \notin g[E] \\ 0 & y \in g[E]. \end{cases}$$