Hence

$$T_a^b(f) \leq g(b) + h(b) - g(a) - h(a). \quad \blacksquare$$

6. Corollary: *If f is of bounded variation on $[a, b]$, then $f'(x)$ exists for almost all x in $[a, b]$.*

Problems

7. a. Let f be of bounded variation on $[a, b]$. Show that for each $c \in (a, b)$ the limit of $f(x)$ exists as $x \to c-$ and also as $x \to c+$. Prove that a monotone function (and hence a function of bounded variation) can have only a countable number of discontinuities. [Hint: If f is monotone, the number of points where $|f(c+) - f(c-)| > 1/n$ is finite.]

b. Construct a monotone function on [0, 1] which is discontinuous at each rational point.

8. a Show that if $a \leq c \leq b$, then $T_a^b = T_a^c + T_c^b$ and that hence $T_a^c \leq T_a^b$.

b. Show that $T_a^b(f + g) \leq T_a^b(f) + T_a^b(g)$, and $T_a^b(cf) = |c|\, T_a^b(f)$.

9. Let $\langle f_n \rangle$ be a sequence of functions on $[a, b]$ that converges at each point of $[a, b]$ to a function f. Then $T_a^b(f) \leq \underline{\lim}\, T_a^b(f_n)$.

10. a. Let f be defined by $f(0) = 0$ and $f(x) = x^2 \sin (1/x^2)$ for $x \neq 0$. Is f of bounded variation on $[-1, 1]$?

b. Let g be defined by $g(0) = 0$ and $g(x) = x^2 \sin (1/x)$ for $x \neq 0$. Is g of bounded variation on $[-1, 1]$?

11. Let f be of bounded variation on $[a, b]$. Show that

$$\int_a^b |f'| \leq T_a^b(f).$$

3 Differentiation of an Integral

If f is an integrable function on $[a, b]$, we define its indefinite integral to be the function F defined on $[a, b]$ by

$$F(x) = \int_a^x f(t)\, dt.$$

In this section we show that the derivative of the indefinite integral of an integrable function is equal to the integrand almost everywhere. We begin by establishing some lemmas.