g real-valued function on the almost everywhere. The deriv-

$$)-f(a).$$

where any two derivates are nsider only the set E where n other combinations of deriset E is the union of the sets

$$> v > D_- f(x)$$

ces to prove that  $m^*E_{u,v} = 0$ . enclose  $E_{u,v}$  in an open set O n  $E_{u,v}$ , there is an arbitrarily ) such that

collection  $\{I_1, ..., I_N\}$  of them  $E_{u,v}$  of outer measure greater intervals, we have

$$\begin{cases} v \sum_{n=1}^{N} h_n \\ < vmO \end{cases}$$

$$< v(s + \epsilon).$$

dpoint of an arbitrarily small d in some  $I_n$  and for which again, we can pick out a finite als such that their union conire greater than  $s-2\epsilon$ . Then

$$)>u\sum k_i$$

$$> u(s-2\epsilon).$$

interval  $I_n$ , and if we sum over

$$f(x_n) - f(x_n - h_n),$$

Sec. 1] Differentiation of Monotone Functions

since f is increasing. Thus

$$\sum_{n=1}^{N} f(x_n) - f(x_n - h_n) \ge \sum_{i=1}^{M} f(y_i + k_i) - f(y_i),$$

and so

$$v(s+\epsilon) > u(s-2\epsilon)$$
.

Since this is true for each positive  $\epsilon$ , we have  $vs \ge us$ . But u > v, and so s must be zero.

This shows that

$$g(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

is defined almost everywhere and that f is differentiable wherever g is finite. Let

$$g_n(x) = n[f(x + 1/n) - f(x)],$$

where we set f(x) = f(b) for  $x \ge b$ . Then  $g_n(x) \to g(x)$  for almost all x, and so g is measurable. Since f is increasing, we have  $g_n \ge 0$ . Hence by Fatou's Lemma

$$\int_{a}^{b} g \le \underline{\lim} \int_{a}^{b} g_{n} = \underline{\lim} n \int_{a}^{b} [f(x+1/n) - f(x)] dx$$

$$= \underline{\lim} \left[ n \int_{b}^{b+1/n} f - n \int_{a}^{a+1/n} f \right]$$

$$= \underline{\lim} \left[ f(b) - n \int_{a}^{a+1/n} f \right]$$

$$\le f(b) - f(a).$$

This shows that g is integrable and hence finite almost everywhere. Thus f is differentiable a.e. and g = f' a.e.

## **Problems**

1. Let f be the function defined by f(0) = 0 and  $f(x) = x \sin(1/x)$  for  $x \neq 0$ . Find  $D^+f(0)$ ,  $D_+f(0)$ ,  $D^-f(0)$ , and  $D_-f(0)$ .

**2. a.** Show that 
$$D^{+}[-f(x)] = -D_{+}f(x)$$
.

**b.** If 
$$g(x) = f(-x)$$
, then  $D^+g(x) = -D_-f(-x)$ .

3. a. If f is continuous on [a, b] and assumes a local maximum at  $c \in (a, b)$ , then

$$D_-f(c) \le D^-f(c) \le 0 \le D_+f(c) \le D^+f(c).$$

**b.** What if f has a local maximum at a or b?

**4.** Prove Proposition 2. [Hint: First show this for a function g for which  $D^+g \ge \varepsilon > 0$ . Apply this to the function  $g(x) = f(x) + \epsilon x$ .]

5. a. Show that  $D^+(f+g) \le D^+f + D^+g$ .

b. State and prove similar inequalities for the other derivates.

**c.** Let f and g be nonnegative and continuous at c. Then

$$D^+(f \cdot g)(c) \le f(c)D^+g(c) + g(c)D^+f(c).$$

**6.** Let f be defined on [a, b] and g a continuous function on  $[\alpha, \beta]$  that is differentiable at  $\gamma$  with  $g(\gamma) = c \ \epsilon \ (a, b)$ .

**a.** If  $g'(\gamma) > 0$ , then  $D^+(f \circ g)(\gamma) = D^+f(c) \cdot g'(\gamma)$ .

**b.** If  $g'(\gamma) < 0$ , then  $D^+(f \circ g)(\gamma) = D_-f(c) \cdot g'(\gamma)$ .

**c.** If  $g'(\gamma) = 0$  and all derivates of f are finite at c, then

$$D^+(f\circ g)(\gamma)=0.$$

## 2 Functions of Bounded Variation

Let f be a real-valued function defined on the interval [a, b], and let  $a = x_0 < x_1 < \cdots < x_k = b$  be any subdivision of [a, b]. Define

$$p = \sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^+$$

$$n = \sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^{-1}$$

$$t = n + p = \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})|,$$

where we use  $r^+$  to denote r, if  $r \ge 0$  and 0, if  $r \le 0$ , and set  $r^- = |r| - r^+$ . We have f(b) - f(a) = p - n. Set

$$P = \sup p$$
,

$$N = \sup n$$

$$T = \sup t$$
,

where we take the suprema over all possible subdivisions of [a, b].