since f is increasing. Thus

$$\sum_{n=1}^{N} f(x_n) - f(x_n - h_n) \geq \sum_{i=1}^{M} f(y_i + k_i) - f(y_i),$$

and so

$$v(s + \epsilon) > u(s - 2\epsilon).$$

Since this is true for each positive ϵ, we have $vs \geq us$. But $u > v$, and so s must be zero.

This shows that

$$g(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

is defined almost everywhere and that f is differentiable wherever g is finite. Let

$$g_n(x) = n[f(x + 1/n) - f(x)],$$

where we set $f(x) = f(b)$ for $x \geq b$. Then $g_n(x) \to g(x)$ for almost all x, and so g is measurable. Since f is increasing, we have $g_n \geq 0$. Hence by Fatou's Lemma

$$\int_a^b g \leq \underline{\lim} \int_a^b g_n = \underline{\lim}\, n \int_a^b [f(x + 1/n) - f(x)]\, dx$$

$$= \underline{\lim} \left[n \int_b^{b+1/n} f - n \int_a^{a+1/n} f \right]$$

$$= \underline{\lim} \left[f(b) - n \int_a^{a+1/n} f \right]$$

$$\leq f(b) - f(a).$$

This shows that g is integrable and hence finite almost everywhere. Thus f is differentiable a.e. and $g = f'$ a.e. ∎

Problems

1. Let f be the function defined by $f(0) = 0$ and $f(x) = x \sin(1/x)$ for $x \neq 0$. Find $D^+f(0)$, $D_+f(0)$, $D^-f(0)$, and $D_-f(0)$.

2. **a.** Show that $D^+[-f(x)] = -D_+f(x)$.

b. If $g(x) = f(-x)$, then $D^+g(x) = -D_-f(-x)$.

3. a. If f is continuous on $[a, b]$ and assumes a local maximum at $c \in (a, b)$, then

$$D_-f(c) \le D^-f(c) \le 0 \le D_+f(c) \le D^+f(c).$$

b. What if f has a local maximum at a or b?

4. Prove Proposition 2. [Hint: First show this for a function g for which $D^+g \ge \varepsilon > 0$. Apply this to the function $g(x) = f(x) + \epsilon x$.]

5. a. Show that $D^+(f + g) \le D^+f + D^+g$.

b. State and prove similar inequalities for the other derivates.

c. Let f and g be nonnegative and continuous at c. Then

$$D^+(f \cdot g)(c) \le f(c)D^+g(c) + g(c)D^+f(c).$$

6. Let f be defined on $[a, b]$ and g a continuous function on $[\alpha, \beta]$ that is differentiable at γ with $g(\gamma) = c \in (a, b)$.

a. If $g'(\gamma) > 0$, then $D^+(f \circ g)(\gamma) = D^+f(c) \cdot g'(\gamma)$.

b. If $g'(\gamma) < 0$, then $D^+(f \circ g)(\gamma) = D_-f(c) \cdot g'(\gamma)$.

c. If $g'(\gamma) = 0$ and all derivates of f are finite at c, then

$$D^+(f \circ g)(\gamma) = 0.$$

2 Functions of Bounded Variation

Let f be a real-valued function defined on the interval $[a, b]$, and let $a = x_0 < x_1 < \cdots < x_k = b$ be any subdivision of $[a, b]$. Define

$$p = \sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^+$$

$$n = \sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^-$$

$$t = n + p = \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})|,$$

where we use r^+ to denote r, if $r \ge 0$ and 0, if $r \le 0$, and set $r^- = |r| - r^+$. We have $f(b) - f(a) = p - n$. Set

$$P = \sup p,$$

$$N = \sup n,$$

$$T = \sup t,$$

where we take the suprema over all possible subdivisions of $[a, b]$.