whenever $x < y$. It is called monotone (or strictly monotone) if f or $-f$ is monotone (or strictly monotone) increasing. Let f be a continuous function on the interval $[a, b]$. Then there is a continuous function g such that $g(f(x)) = x$ for all $x \in [a, b]$ if and only if f is strictly monotone. In this case we also have $f(g(y)) = y$ for each y between $f(a)$ and $f(b)$. A function f which has a continuous inverse is called a *homeomorphism* (between its domain and its range).

47. A continuous function φ on $[a, b]$ is called *polygonal* (or *piecewise linear*) if there is a subdivision $a = x_0 < x_1 < \cdots < x_n = b$ such that φ is linear on each interval $[x_i, x_{i+1}]$. Let f be an arbitrary continuous function on $[a, b]$ and ϵ a positive number. Show that there is a polygonal function φ on $[a, b]$ with $|f(x) - \varphi(x)| < \epsilon$ for all $x \in [a, b]$.

48. Let x be a real number in $[0, 1]$ with the ternary expansion $\langle a_n \rangle$ (cf. Problem 22). Let $N = \infty$ if none of the a_n are 1, and otherwise let N be the smallest value of n such that $a_n = 1$. Let $b_n = \frac{1}{2}a_n$ for $n < N$ and $b_N = 1$. Show that

$$\sum_{n=1}^{N} \frac{b_n}{2^n}$$

is independent of the ternary expansion of x (if x has two expansions) and that the function f defined by setting

$$f(x) = \sum_{n=1}^{N} \frac{b_n}{2^n}$$

is a continuous, monotone function on the interval $[0, 1]$. Show that f is constant on each interval contained in the complement of the Cantor ternary set (Problem 37), and that f maps the Cantor ternary set *onto* the interval $[0, 1]$. (This function is called the *Cantor ternary function*.)

49. *Limit superior of a function of a real variable.* Let f be a real (or extended real) valued function defined for all x in an interval containing y. We define

$$\overline{\lim_{x \to y}}\, f(x) = \inf_{\delta > 0} \sup_{0 < |x - y| < \delta} f(x)$$

$$\overline{\lim_{x \to y+}}\, f(x) = \inf_{\delta > 0} \sup_{0 < x - y < \delta} f(x)$$

with similar definitions for $\underline{\lim}$.

a. $\overline{\lim}_{x \to y} f(x) \le A$ if and only if, given $\epsilon > 0$, there is a $\delta > 0$ such that for all x with $0 < |x - y| < \delta$ we have $f(x) \le A + \epsilon$.

b. $\overline{\lim}_{x \to y} f(x) \ge A$ if and only if, given $\epsilon > 0$ and $\delta > 0$, there is an x such that $0 < |x - y| < \delta$ and $f(x) \ge A - \epsilon$.

c. $\varliminf_{x\to y} f(x) \le \varlimsup_{x\to y} f(x)$ with equality (for $\overline{\lim} f \ne \pm\infty$) if and only if $\lim_{x\to y} f(x)$ exists.

d. If $\varlimsup_{x\to y} f(x) = A$ and $\langle x_n\rangle$ is a sequence with $x_n \ne y$ such that $y = \lim x_n$, then $\overline{\lim} f(x_n) \le A$.

e. If $\varlimsup_{x\to y} f(x) = A$, then there is a sequence $\langle x_n\rangle$ with $x_n \ne y$ such that $y = \lim x_n$ and $A = \lim f(x_n)$.

f. For a real number l we have $l = \lim_{x\to y} f(x)$ if and only if $l = \lim f(x_n)$ for every sequence $\langle x_n\rangle$ with $x_n \ne y$ and $y = \lim x_n$.

50. *Semicontinuous functions.* An extended real-valued function f is called *lower semicontinuous* at the point y if $f(y) \ne -\infty$ and $f(y) \le \varliminf_{x\to y} f(x)$. Similarly, f is called *upper semicontinuous* at y if $f(y) \ne +\infty$ and $f(y) \ge \varlimsup_{x\to y} f(x)$. We say that f is lower (upper) semicontinuous on an interval if it is lower (upper) semicontinuous at each point of the interval. The function f is upper semicontinuous if and only if the function $-f$ is lower semicontinuous.

a. Let $f(y)$ be finite. Prove that f is lower semicontinuous at y if and only if given $\epsilon > 0$, $\exists\delta > 0$ such that $f(y) \le f(x) + \epsilon$ for all x with $|x - y| < \delta$.

b. A function f is continuous (at a point or in an interval) if and only if it is both upper and lower semicontinuous (at the point or in the interval).

c. Show that a real-valued function f is lower semicontinuous on (a, b) if and only if the set $\{x : f(x) > \lambda\}$ is open for each real number λ.

d. Show that if f and g are lower semicontinuous functions, so are $f \vee g$ and $f + g$.

e. Let $\langle f_n\rangle$ be a sequence of lower semicontinuous functions. Show that the function f defined by $f(x) = \sup_n f_n(x)$ is also lower semicontinuous.

f. A real-valued function φ defined on an interval $[a, b]$ is called a **step function** if there is a partition $a = x_0 < x_1 < \cdots < x_n = b$ such that for each i the function φ assumes only one value in the interval (x_i, x_{i+1}). Show that a step function φ is lower semicontinuous iff $\varphi(x_i)$ is less than or equal to the smaller of the two values assumed in (x_{i-1}, x_i) and (x_i, x_{i+1}).

g. A function f defined on an interval $[a, b]$ is lower semicontinuous if and only if there is a monotone increasing sequence $\langle \varphi_n\rangle$ of lower semicontinuous step functions on $[a, b]$ such that for each $x \in [a, b]$ we have $f(x) = \lim \varphi_n(x)$.

h. A function f defined on $[a, b]$ is lower semicontinuous if and only