# Quiz # 4.

In solutions below some definitions are omitted.

**Problem 2.** Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x, & x \in (-\infty, 2], \\ 2, & x \in (2, 3], \\ 4, & x \in (3, \infty). \end{cases}$$

Let $\lambda_f$ be the corresponding Lebesgue-Stieltjes measure. Find $\lambda_f(\{0\})$, $\lambda_f((-1,1))$, $\lambda_f((2,4])$.

**Solution.** Since $f(x) = x$ on $(-\infty, 2]$, $\lambda_f$ coincides with the Lebesgue measure on $(-\infty, 2]$. Indeed, if $A \subset (-\infty, 2]$ is covered by intervals $I_k = (a_k, b_k]$ then $A$ is also covered by intervals $J_k = I_k \cap (-\infty, 2]$ and $\alpha_f(J_k) \leq \alpha(I_k) = f(b_k) - f(a_k)$ for every $k$. It shows that one can consider coverings by $I_k$'s with $b_k \leq 2$ for every $k$. But on such $I_k$'s, $\alpha(I_k)$ coincide with volume/length $v(I_k) = b_k - a_k$. Thus, for $A \subset (-\infty, 2]$ the Lebesgue-Stieltjes measure coincides with the Lebesgue measure. We also know that on intervals the Lebesgue measure is equal to the length, thus $\lambda_f(\{0\}) = 0$, $\lambda_f((-1,1)) = 2$.

We show now that $\lambda_f((2,4]) = 0$. Let $A_1 = (2,3]$ and $A_n = (3 + 1/n, 4]$ for $n \geq 2$. Then

$$(2,4] \subset \bigcup_{n=1}^{\infty} A_n$$

and

$$\alpha_f(A_1) = f(3) - f(2) = 0, \quad \alpha_f(A_n) = f(4) - f(3 + 1/n) = 0, \quad \text{for every } n \geq 2.$$

Thus

$$0 \leq \lambda_f((2,4]) \leq \sum_{n=1}^{\infty} \alpha_f(A_n) = 0,$$

which implies the desired result. $\square$

**Answer.** $\lambda_f(\{0\}) = 0$, $\lambda_f((-1,1)) = 2$, $\lambda_f((2,4]) = 0$.

**Remark.** Another way to see that $\lambda_f(\{0\}) = 0$ and $\lambda_f((-1,1)) = 2$ is

**a.** Consider $A = (-1/n, 0]$. Clearly, $\{0\} \subset A_n$ and, thus

$$0 \leq \lambda_f(\{0\}) \leq \lambda_f(A) \leq \alpha_f(A) = f(0) - f(-1/n) = 1/n.$$

Sending $n$ to infinity we obtain the result.

**b.** Assume $I = (-1,1)$ is covered by intervals $I_k = (a_k, b_k]$. Without loss of generality we can assume that $b_k \leq 2$ for every $k$ (otherwise consider intervals $J_k = I_k \cap (-\infty, 2]$). Note also that $I$ is covered by $A_k = [a_k, b_k]$. By a lemma in the class it implies

$$2 = v(I) \leq \sum_{n=1}^{\infty} v(A_n) = \sum_{n=1}^{\infty} (b_n - a_n) = \sum_{n=1}^{\infty} \alpha_f(A_n),$$

where $v(\cdot)$ denotes the volume (length). It shows that $2 \leq \lambda_f((-1,1))$. On the other hand,

$$\lambda_f((-1,1)) \leq \lambda_f((-1,1]) \leq \alpha_f((-1,1]) = 2.$$