Quiz # 3.

In solutions below some definitions are omitted.

Problem 2. Let (X, ρ) be a complete metric space. Let $\{F_n\}$ be a decreasing sequence of non-empty closed subsets of X such that $\mathrm{diam} F_n \to 0$ as $n \to \infty$. Show that $\lim_{n\to\infty} F_n$ is a singleton.

Solution. Since each F_n is non-empty, for every $n \geq 1$ we can choose $x_n \in F_n$. Now, for every $m \geq n$ we have $F_m \subset F_n$. It implies $x_n, x_m \in F_n$, hence

$$\rho(x_n, x_m) \leq \mathrm{diam} F_n$$

for every $m \geq n$. Since $\mathrm{diam} F_n \to 0$ as $n \to \infty$, we obtain

$$\rho(x_n, x_m) \to 0 \quad \text{as} \quad n, m \to \infty.$$

It shows that $\{x_n\}_{n\geq 1}$ is a fundamental sequence. Since X is complete, $\{x_n\}_{n\geq 1}$ is convergent, *i.e.*, there exists $x_0 \in X$, such that $x_n \to x_0$ as $n \to \infty$. Using again that $F_m \subset F_n$ for every $m \geq n$, we notice that the sequence $\{x_m\}_{m\geq n} \subset F_n$ for every n. Since F_n is closed for every n, each F_n contains all limit points. Therefore $x_0 \in F_n$ for every n (clearly, x_0 is the limit of $\{x_m\}_{m\geq n}$ for every n). It implies that

$$x_0 \in \bigcap_{n\geq 1} F_n = \lim_{n\to\infty} F_n.$$

In other words $\lim_{n\to\infty} F_n \neq \emptyset$.

Now assume $\lim_{n\to\infty} F_n$ contains two different points, say x, y. Then $x, y \in F_n$ for every n, thus

$$\mathrm{diam} F_n \geq \rho(x, y) > 0$$

for every n. It contradicts to the fact that $\mathrm{diam} F_n \to 0$. Therefore, $\lim_{n\to\infty} F_n$ consists of exactly one point, that is, $\lim_{n\to\infty} F_n$ is a singleton. $\square$

Problem 3. Let ϕ be an outer measure on X and $E \subset X$. Assume $\phi(E) = 0$. Show that E is measurable.

Solution. Let $A \subset X$. Since ϕ is monotone, we have $\phi(A \cap E) \leq \phi(E) = 0$ and $\phi(A \setminus E) \leq \phi(A)$. It implies

$$\phi(A \cap E) + \phi(A \setminus E) \leq \phi(A).$$

Since the opposite inequality is trivial (because ϕ is subadditive), we obtain

$$\phi(A \cap E) + \phi(A \setminus E) = \phi(A).$$

It shows that E is a measurable set. $\square$