Quiz # 2.

In solutions below some definitions are omitted.

Problem 1.
a. Give definitions of a countable set, of a fundamental sequence.
b. Is any fundamental sequence convergent? (answer yes/no only.)
c. Is any convergent sequence fundamental? (answer yes/no only.)

Answer.
b. No, not every fundamental sequence is convergent.
c. Yes, every convergent sequence fundamental.

Problem 2. Let A, B, C, D be nonempty sets such that card $A =$ card C, card $B =$ card D. Show that card $A^B =$ card C^D (start with the definition of A^B).

Solution. Recall

$$A^B = \{f : B \to A\}, \qquad C^D = \{f : D \to C\}.$$

Since card $A =$ card C, card $B =$ card D there exist bijections $F : A \to C$ and $G : B \to D$. Since F and G are bijections, there are inverse functions $F^{-1} : C \to A$ and $G^{-1} : D \to B$. We define a function $H : A^B \to C^D$ by

$$H(f) = F \circ f \circ G^{-1},$$

for every $f \in A^B$; that is

$$(H(f))(x) = F(f(G^{-1}(x)))$$

for every $f \in A^B$ and every $x \in D$. (Note that for every $f \in A^B$ the value $H(f)$ should be in C^D, thus $H(f)$ is a function $C \to D$. To define it, we should explain how it works at elements of D.)

By the definition of the composition, H is well-defined (for every $f \in A^B$, $H(f)$ is indeed an element of C^D). We show that H is a bijection.

Assume $f, g \in A^B$ be such that $f \neq g$. Since f, g are functions $B \to A$, it means that there exist $b \in B$ such that $f(b) \neq g(b)$. Since F is an injection we observe that $F(f(b)) \neq F(g(b))$. Now consider $x = G(b) \in D$. We have

$$(H(f))(x) = F(f(G^{-1}(x))) = F(f(b)) \quad \text{and} \quad (H(g))(x) = F(g(G^{-1}(x))) = F(g(b)).$$

It shows that $(H(f))(x) \neq (H(g))(x)$, which in particular means that $H(f)$ and $H(g)$ are different functions, i.e. $H(f) \neq H(g)$. It proves that H is an injection.

Finally, let $g \in C^D$, that is $g : D \to C$. Consider function $f = F^{-1} \circ g \circ G$, i.e. $f(b) = F^{-1}(g(G(b)))$ for every $b \in B$. Clearly, f is a function $B \to A$, i.e. $f \in A^B$. We have also for every $x \in C$

$$(H(f))(x) = F(f(G^{-1}(x))) = F(F^{-1}(g(G(G^{-1}(x))))) = F(F^{-1}(g(x))) = g(x).$$

It means that $H(f) = g$. So for every $g \in C^D$ there exists $f \in A^B$ such that $H(f) = g$. It means that H is a surjection.

We proved that there exists a bijection $A^B \to C^D$, which means card $A^B =$ card C^D. $\square$