Quiz # 1.

In solutions below definitions are omitted.

Problem 1. Let $\{E_i\}_i^\infty$ be a sequence of sets. Give definitions of $\liminf_{i\to\infty} E_i$, $\limsup_{i\to\infty} E_i$, and $\lim_{i\to\infty} E_i$. Show that $\liminf_{i\to\infty} E_i \subset \limsup_{i\to\infty} E_i$.

Solution. Let

$$x \in \liminf_{i\to\infty} E_i = \bigcup_{k=1}^{\infty} \bigcap_{i=k}^{\infty} E_i.$$

Then, by the definition of union, there exists m such that

$$x \in \bigcap_{i=m}^{\infty} E_i.$$

Hence, by the definition of intersection, for every $i \geq m$ we have $x \in E_i$. Therefore, for every $k \geq 1$ we can find $i \geq k$ (for example $i = \max\{k, m\}$) such that $x \in E_i$. It implies that for every $k \geq 1$

$$x \in \bigcup_{i=k}^{\infty} E_i.$$

Since the last inclusion holds for every $k \geq 1$ we observe

$$x \in \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} E_i = \limsup_{i\to\infty} E_i.$$

$\square$

Problem 2. Provide definitions of a relation on a set X, a reflexive relation, a symmetric relation, a transitive relation, an antisymmetric relation. Let R be a relation on $\mathbb{R}$ defined by $(x, y) \in R$ if (and only if) $x - y = 1$. Is R reflexive? Is R symmetric? Is R transitive? Is R antisymmetric?

Solution. **a.** Clearly, $(0, 0) \notin R$. Thus R is not reflexive.

b. Clearly $(1, 0) \in R$, but $(0, 1) \notin R$. Thus R is not symmetric.

c. Clearly $(2, 1) \in R$ and $(1, 0) \in R$, but $(2, 0) \notin R$. Thus R is not transitive.

d. Assume that R is not antisymmetric. Then there exists $x, y \in \mathbb{R}^n$ such that $(x, y) \in R$, $(y, x) \in R$, and $x \neq y$. In this case, by the definition of R we have $x - y = 1$ and $y - x = 1$, which is impossible. We got a contradiction. It shows that R is antisymmetric.

Another solution: Let $(x, y) \in R$, $(y, x) \in R$. It means $x - y = 1$ and $y - x = 1$. Hence, $x - y = y - x$. It implies, $2x = 2y$, that is $x = y$. Therefore R is antisymmetric. $\square$

Remark. Note, to show that R is not transitive, it is not enough to say that $(x, y) \in R$ and $(y, z) \in R$ imply $(x, y) \notin R$. In addition, one need to show that R contains such two pairs (x, y) and (y, z). Indeed, consider relation on $\mathbb{R}$ defined by

$$R = \{(x, y) \mid x < 0,\ y > 0\}.$$

Then R is transitive and also has property "$(x, y) \in R$ and $(y, z) \in R$ imply that $(x, y) \notin R$".

Similarly, to show that R is not symmetric, it is not enough to say that $(x, y) \in R$ implies that $(y, x) \notin R$. In addition, one need to show that R contains at least one pair (x, y), that is to show that R is not empty. Indeed, if $R = \emptyset$ then R is symmetric and also has property "$(x, y) \in R$ implies that $(y, x) \notin R$".