Practice exam

1. (15 pt) Test the series for convergence or divergence

a. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{e\,n}{n+1}\right)$ **b.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n\,(n+2)}{n}$ **c.** $\displaystyle\sum_{n=1}^{\infty} \tan(2/n)$

2. (6 pt) Is the following series convergent? If yes, find its sum.

$$\sum_{n=1}^{\infty} \left(3^{-n} + (-7)^n\, 2^{-3n+1}\right)$$

3. (10 pt) Find the radius of convergence and the interval of convergence of the following series

$$\sum_{n=1}^{\infty} \frac{(-2)^n\,(x+3)^n}{n}.$$

4. (10 pt) Using the definition, find the Maclaurin series of $f(x) = x\sin x$.

5. (12 pt) Let T be the triangle with the vertices $(-1,1,1)$, $(2,1,0)$, $(3,0,1)$. Find the plane containing T. Find the area of T.

6. (13 pt) Find the angle between two planes given by $2x + y + z = 4$ and $x + 2y - z = 2$. Are planes orthogonal? Are they parallel? Find their intersection.

7. (17 pt) Find the points of local minima, maxima and saddle points of the function

$$f(x,y) = x^3 + 9xy + y^3.$$

8. (17 pt) Use the Lagrange multipliers to find the minimum and maximum values of the function

$$f(x,y) = x^2 - y + 2y^2$$

subject to the constraint $x^2 + y^2 = 1$.

Solutions

1. a. Let $a_n = \ln\left(\frac{e\,n}{n+1}\right)$. We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(\frac{e\,n}{n+1}\right) = \ln\left(\lim_{n\to\infty}\frac{e\,n}{n+1}\right) = \ln(e) = 1.$$

Since $\lim_{n\to\infty} a_n \neq 0$, then by the divergence test this series is divergent.

b. Let $a_n = \frac{(-1)^n\,(n+2)}{n}$. We have $\lim_{n\to\infty} |a_n| = 1 \neq 0$. By the test of divergence, this series diverges.

c. Let $a_n = \tan(2/n) = \frac{\sin(2/n)}{\cos(2/n)}$. Let $b_n = 2/n$. We have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(2/n)}{2/n} \cdot \frac{1}{\cos(2/n)} = 1,$$

where we used that $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$. By the limit comparison test, $\sum a_n$ and $\sum b_n$ have the same behavior. Since $\sum b_n$ is divergent (Harmonic series), then $\sum a_n$ is also divergent.

Answer. All 3 series are divergent.

2. Let $a_n = 3^{-n} + (-7)^n\, 2^{-3n+1}$. Denote $b_n = 3^{-n}$ and $c_n = (-7)^n\, 2^{-3n+1}$. We will treat each series separately.

Note that $\sum_{n\geq 1} b_n$ is a geometric series with parameters $a = 1$ and $r = 1/3$. Since $|r| < 1$, it is convergent and its sum equals

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} ar^n = \frac{ar}{1-r} = \frac{1}{2}.$$

Now we have

$$c_n = (-7)^n\, 2^{-3n+1} = 2\left(-\frac{7}{8}\right)^n,$$

hence $\sum_{n\geq 1} c_n$ is a geometric series with parameters $a = 2$ and $r = -7/8$. Since $|r| < 1$, it is convergent and its sum is equal to

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} ar^n = \frac{ar}{1-r} = \frac{-14/8}{1+7/8} = -\frac{14}{15}.$$

Since both $\sum_{n\geq 1} b_n$ and $\sum_{n\geq 1} c_n$ are convergent, then $\sum_{n\geq 1} a_n$ is also convergent and we have

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} c_n = \frac{1}{2} - \frac{14}{15} = -\frac{13}{30}.$$

Answer. The series converges to $-13/30$.

3. Let $a_n = \frac{(-2)^n\,(x+3)^n}{n}$. We will use the ratio test.

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 2|x+3| \cdot \lim_{n\to\infty} \frac{n}{n+1} = 2|x+3|.$$

By the ratio test: if $2|x+3| < 1$, then the series is convergent and if $2|x+3| > 1$ the series is divergent. This means that if $x \in (-7/2, -5/2)$, the series is divergent. The radius of convergence is $1/2$. We still have to check the end points:

-If $x = -7/2$, then $a_n = 1/n$. Therefore $\sum a_n$ is the Harmonic series which is divergent by the integral test.

-If $x = -5/2$, then $a_n = (-1)^n/n$. Therefore $\sum a_n$ is the alternating Harmonic series which is convergent by the alternating series test.

As a conclusion, the interval of convergence is $(-7/2, -5/2]$ and the radius of convergence is $1/2$.

Answer. The radius is $1/2$, the interval is $(-7/2, -5/2]$.

4. We have to calculate the derivatives of f at 0. We have:

-$f(x) = x\sin(x)$, thus $f(0) = 0$.
-$f'(x) = \sin(x) + x\cos(x)$, thus $f'(0) = 0$.
-$f''(x) = 2\cos(x) - x\sin(x) = 2\cos(x) - f(x)$, thus $f''(0) = 2$.
-$f^{(3)}(x) = -2\sin(x) - f'(x)$, thus $f^{(3)}(0) = 0$.
-$f^{(4)}(x) = -2\cos(x) - f''(x) = -4\cos(x) + f(x)$, thus $f^{(4)}(0) = -4$.
We see that all odd derivatives are zero. We have

$$f^{(2n+1)}(0) = 0, \quad f^{(2n)}(0) = (-1)^{n+1}(2n).$$

The Maclaurin series of f is given by

$$\sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{(2n)!} x^{2n} + \sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)}{(2n+1)!} x^{2n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)!} x^{2n}.$$

Answer.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)!} x^{2n}.$$

5. Let $A(-1,1,1)$, $B(2,1,0)$ and $C(3,0,1)$. A normal vector to the plane formed by A, B, C is $\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$. We have

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 0 & -1 \\ 4 & -1 & 0 \end{vmatrix} = -\vec{i} - 4\vec{j} - 3\vec{k} = (-1, -4, -3).$$

Now a point $M(x,y,z)$ belongs to this plane if and only if $\overrightarrow{AM} \cdot \vec{n} = 0$. This means that $(x+1)\cdot(-1) + (y-1)\cdot(-4) + (z-1)\cdot(-3) = 0$. Therefore, the equation of the plane is given by $x + 4y + 3z = 6$.

The area of the triangle T is given by $\frac{|\overrightarrow{AB} \times \overrightarrow{AC}|}{2} = \frac{\sqrt{26}}{2}$.

Answer. The plane is given by $x + 4y + 3z = 6$, the area is $\sqrt{26}/2$.

6. Let $\vec{n_1} = (2,1,1)$ be the normal vector to the first plane and $\vec{n_2} = (1,2,-1)$ the normal vector to the second plane. Denote θ be the angle between the two planes. We have

$$\cos(\theta) = \frac{|\vec{n_1} \cdot \vec{n_2}|}{|\vec{n_1}| \cdot |\vec{n_2}|} = \frac{3}{\sqrt{6} \cdot \sqrt{6}} = \frac{1}{2}.$$

Therefore the angle θ is $\pi/3$. We deduce that these two planes are neither parallel nor orthogonal.
First method: In order to find their intersection, we may solve the system

$$\begin{cases} 2x + y + z = 4 \\ x + 2y - z = 2 \end{cases}$$

Put $x = t$ and solve the system

$$\begin{cases} y + z = 4 - 2t \\ 2y - z = 2 - t \end{cases}$$

Adding the two equations, we get $y = 2 - t$ and $z = 2 - t$. Therefore the equation of the intersection line is given by

$$\begin{cases} x = t \\ y = 2 - t \\ z = 2 - t \end{cases}$$

Second method: The direction vector of the intersection line is given by $\vec{v} = \vec{n_1} \times \vec{n_2}$. We have

$$\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = -3\vec{i} + 3\vec{j} + 3\vec{k} = (-3, 3, 3).$$

We still need to find a common point on both planes. For that take $x = 0$ and solve the system

$$\begin{cases} y + z = 4 \\ 2y - z = 2 \end{cases}$$

By adding the two equations we get $y = 2$ and after replacing we get $z = 2$. Therefore the point $A(0, 2, 2)$ is a point of the intersection line. Now $M(x, y, z)$ belongs to the intersection line if and only if $\overrightarrow{AM} = t\vec{v}$. This means that the equation of the intersection line is given by

$$\begin{cases} x = -3t \\ y = 3t + 2 \\ z = 3t + 2 \end{cases}$$

Answer. The angle is $\pi/3$; the planes are neither parallel nor orthogonal; the intersection is the line given by

$$\begin{cases} x = -3t \\ y = 3t + 2 \\ z = 3t + 2. \end{cases}$$

7. We need first to find the critical points. First calculate partial derivatives.

$$f_x(x, y) = 3x^2 + 9y; \quad f_y(x, y) = 3y^2 + 9x.$$

We have a critical point when $f_x(x, y) = f_y(x, y) = 0$. This means that $x^2 = -3y$ and $y^2 = -3x$. Replacing the second equation in the first we get $y^4 = -27y$ which means that either $y = 0$ or $y = -3$. This means that $(0, 0)$ and $(-3, -3)$ are the critical points of f.

We may now use the second derivative test in order to classify the critical points. Let us first calculate the second order derivatives.

$$f_{xx}(x, y) = 6x; \quad f_{yy}(x, y) = 6y; \quad f_{xy}(x, y) = 9.$$

We have

$$D(a, b) = \begin{vmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{xy}(a, b) & f_{yy}(a, b) \end{vmatrix}$$

Since $D(0, 0) = -81 < 0$, then by the second derivative test, $(0, 0)$ is a saddle point.
Since $D(-3, -3) = 243 > 0$ and $f_{xx}(-3, -3) = -18 < 0$ then by the second derivative test $(-3, -3)$ is a local maximum point. This means that f has a local minimum equal to $f(-3, -3) = 27$ at the point $(-3, -3)$.

Answer. $(0, 0)$ is a saddle point, $(-3, -3)$ is a local maximum point.

8. Let $g(x) = x^2 + y^2$. Using Lagrange multipliers, the extremum is attained when $\nabla f = \lambda \nabla g$. Therefore we get the following system

$$\begin{cases} f_x(x, y) = \lambda g_x(x, y) \\ f_y(x, y) = \lambda g_y(x, y) \\ g(x, y) = 1 \end{cases}$$

After calculating we have the following system

$$\begin{cases} x = \lambda x \\ 4y - 1 = 2\lambda y \\ x^2 + y^2 = 1 \end{cases}$$

Looking at the first equation, we have either $x = 0$ or $\lambda = 1$.

-If $x = 0$, then using the third equation we get $y = \pm 1$. We have $f(0, 1) = 1$ and $f(0, -1) = 3$.

-If $\lambda = 1$, using the second equation we get $y = 1/2$ and replacing in the third equation we get $x = \pm\sqrt{3}/2$. We have $f(\pm\sqrt{3}/2, 1/2) = 3/4$.

We conclude that 3 is the maximum of f attained at $(0, -1)$ and $3/4$ is the minimum of f attained at the points $(\sqrt{3}/2, 1/2)$ and $(-\sqrt{3}/2, 1/2)$.

Answer. The maximum is 3, the minimum is $3/4$.