Quiz # 6.

Problem 1. Find the tangent plane to the surface $z = 4^x \ln(xy)$ at $(1, e, 4)$.

Solution. Let $f(x, y) = 4^x \ln(xy)$. Then

$$f_x = 4^x \ln 4 \ln(xy) + \frac{4^x}{x} \quad \text{and} \quad f_y = \frac{4^x}{y}.$$

In general the tangent plane is given by $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$. In our case $(x_0, y_0, z_0) = (1, e, 4)$, therefore

$$z - 4 = f_x(1, e)(x-1) + f_y(1, e)(y-e)(4 \ln 4 + 4)(x-1) + \frac{4}{e}(y-e) = (4 \ln 4 + 4)x + \frac{4}{e} y - 4 \ln 4 - 8,$$

that is $(4 \ln 4 + 4)x + (4/e)y - z = 4 \ln 4 + 4$.

Answer. $(4 \ln 4)x + (4/e)y - z = 4 \ln 4 + 4$.

Problem 2. Find the points of local minima, maxima and saddle points of the function

$$f(x, y) = 2x \cos y - x^2, \qquad x \in \mathbb{R},\ y \in [-2, 4].$$

Solution. First, to find critical points of f, we solve $f_x(x, y) = f_y(x, y) = 0$, that is

$$\begin{cases} 2 \cos y - 2x = 0 \\ -2x \sin y = 0. \end{cases}$$

From the second equation we have either $x = 0$ or $\sin y = 0$. When $x = 0$, from the first equation $\cos y = 0$. Since $y \in [-2, 4]$, we observe $y = \pi/2$ or $y = -\pi/2$. Thus first two critical points are $(0, \pi/2)$ and $(0, -\pi/2)$. When $\sin y = 0$, we have $y = 0$ or $y = \pi$ (we used again that $y \in [-2, 4]$). If $y = 0$ then $\cos y = 1$, so from the first equation $x = 1$. If $y = \pi$ then $\cos y = -1$, so from the first equation $x = -1$. Therefore we have two more critical points $(1, 0)$ and $(-1, \pi)$. Now we find the second derivatives and the corresponding determinant. $f_{xx} = -2$, $f_{yy} = -2x \cos y$, $f_{xy} = -2 \sin y$. Therefore,

$$D(x, y) = \begin{vmatrix} f_{xx}(x, y) & f_{xy}(x, y) \\ f_{xy}(x, y) & f_{xx}(y, y) \end{vmatrix} = f_{xx}(x, y) f_{yy}(x, y) - f_{xy}^2(x, y) = 4(x \cos y - \sin^2 x).$$

At points $(0, \pi/2)$ and $(0, -\pi/2)$ we have $D(0, \pi/2) = D(0, -\pi/2) = -4 < 0$, therefore they are saddle points.

At points $(1, 0)$ and $(-1, \pi)$ we have $D(1, 0) = D(-1, \pi) = 4 > 0$ and $f_{xx}(1, 0) = f_{xx}(-1, \pi) = -2 < 0$, therefore they are points of local maximum.

Answer. f has local maxima at $(1, 0)$ and $(-1, \pi)$ and saddle points at $(0, \pi/2)$ and $(0, -\pi/2)$.