Quiz # 6.

Problem 1. Find the tangent plane to the surface $z = x\,e^{xy}$ at $(2,0,2)$.

Solution. Let $f(x,y) = x\,e^{xy}$. Then

$$f_x = e^{xy} + xy\,e^{xy} \qquad \text{and} \qquad f_y = x^2\,e^{xy}.$$

In general the tangent plane is given by $z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$. In our case $(x_0,y_0,z_0) = (2,0,2)$, therefore

$$z - 2 = f_x(2,0)(x-2) + f_y(2,0)y = x - 2 + 4y,$$

that is $x + 4y - z = 0$.

Answer. $x + 4y - z = 0$.

Problem 2. Using the chain rule find

a. $\dfrac{dz}{dt}$, where $z = \sqrt{x^2+y}$, $x = \cos t$, $y = \sin t$.

b. $\dfrac{\partial z}{\partial s}$, where $z = x\,\ln y$, $x = e^{t+2s}$, $y = t/s$.

Solution. **a.**

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \frac{x}{\sqrt{x^2+y}}(-\sin t) + \frac{1}{2\sqrt{x^2+y}}\cos t = -\frac{\cos t\,\sin t}{\sqrt{\cos^2 t + \sin t}} + \frac{\cos t}{2\sqrt{\cos^2 t + \sin t}}.$$

b.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (\ln y)\,2e^{t+2s} + \frac{x}{y}\left(-\frac{t}{s^2}\right) = 2e^{t+2s}\,\ln\frac{t}{s} - \frac{e^{t+2s}}{s}.$$

Answer.

a. $\dfrac{\cos t - 2\cos t\,\sin t}{2\sqrt{\cos^2 t + \sin t}}$ **b.** $2e^{t+2s}\,\ln\dfrac{t}{s} - \dfrac{e^{t+2s}}{s}$.

Problem 3. Let $f(x,y,z) = x\,(\ln y)^2 + z$. Find the gradient of f. Find the directional derivative of f at $(0,e,0)$ in the direction of $\vec{u} = (-1/2, 1/\sqrt{2}, 1/2)$.

Solution.

$$\nabla f = (f_x, f_y, f_z) = ((\ln y)^2, (2x\,\ln y)/y, 1).$$

The directional derivative at $(0,e,0)$ is

$$D_{\vec{u}}f = \nabla f(0,e,0)\cdot\vec{u} = (1,0,1)\cdot(-1/2, 1/\sqrt{2}, 1/2) = -1/2 + 1/2 = 0.$$

Answer. $\nabla f = ((\ln y)^2, (2x\,\ln y)/y, 1)$, $D_{\vec{u}}f = 0$.