Quiz # 4.

Problem 1. Find the area of the triangle with vertices $A = (0, 0, 0)$, $B = (1, 1, 3)$, $C = (2, -3, 1)$.

Solution. We know that the area is half of the length of the cross product of the vectors $\vec{AB} = (1, 1, 3)$ and $\vec{AC} = (2, -3, 1)$. We have

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 3 \\ 2 & -3 & 1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 1 & 3 \\ -3 & 1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = (10, 5, -5).$$

Thus the area is

$$\frac{1}{2} \left| \vec{AB} \times \vec{AC} \right| = \frac{1}{2} \left| (10, 5, -5) \right| = \frac{1}{2} 5 \left| (2, 1, -1) \right| = \frac{5}{2} \sqrt{4 + 1 + 1} = \frac{5\sqrt{6}}{2}.$$

Answer. The area is $5\sqrt{6}/2$.

Problem 3. Find the intersection of two planes $x + y + 3z = 0$ and $2x - 3y + z = -5$.

Solution. Normal vectors to planes are determined by coefficients in front of x, y, z, that is, in our case, $\vec{n}_1 = (1, 1, 3)$, $\vec{n}_2 = (2, -3, 1)$. If a line ℓ lies in both planes then it is orthogonal to both $\vec{n}_1$ and $\vec{n}_2$. Therefore ℓ is parallel to their cross product of $\vec{n}_1$ and $\vec{n}_2$. By Problem 1, we observe $\vec{n}_1 \times \vec{n}_2 = (10, 5, -5)$. Thus ℓ is parallel to $(10, 5, -5)$. Now we find a point $P_0 = (x_0, y_0, z_0)$ on ℓ. We assume that $z_0 = 0$. As P_0 belongs to both planes (and $z_0 = 0$) we have

$$\begin{cases} x_0 + y_0 = 0 \\ 2x_0 - 3y_0 = -5. \end{cases}$$

From the first equation we have $y_0 = -x_0$, plugging this into the second one we observe $5x_0 = -5$, which implies $x_0 = -1$ and $y_0 = 1$. Therefore the point $P_0 = (-1, 1, 0)$ belongs to the line ℓ, so we obtain the answer.

Answer. The intersection is the line

$$\begin{cases} x = -1 + 10t \\ y = 1 + 5t \\ z = -5t, \end{cases} \qquad t \in \mathbb{R}^n$$

or

$$\frac{x + 1}{2} = y - 1 = -z$$

(both answers are acceptable).