Quiz # 4.

Problem 1. Find the area of the triangle with vertices $A = (1,1,1)$, $B = (2,2,2)$, $C = (-1,1,2)$.

Solution. We know that the area is half of the length of the cross product of the vectors $\vec{AB} = (2,2,2) - (1,1,1) = (1,1,1)$ and $\vec{AC} = (-1,1,2) - (1,1,1) = (-2,0,1)$. We have

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ -2 & 0 & 1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} = (1,-3,2).$$

Thus the area is

$$\frac{1}{2} = \frac{1}{2} \left| \vec{AB} \times \vec{AC} \right| = \frac{1}{2} \left| (1,-3,2) \right| = \frac{1}{2} \sqrt{1+9+4} = \frac{1}{2} \sqrt{14} = \sqrt{7/2}.$$

Answer. The area is $\sqrt{7/2}$.

Problem 2. Find the intersection between the plane $x + y + z = 1$ and the line

$$x = t \qquad y = 1 - 2t \qquad z = 2 + 3t.$$

Solution. If a point $P_0 = (x_0, y_0, z_0)$ belongs to the intersection then (x_0, y_0, z_0) belongs to the plane, that is $x_0 + y_0 + z_0 = 1$, and simultaneously P_0 belongs to the line, that is for some value t_0 we have $x_0 = t_0$, $y_0 = 1 - 2t_0$, $z_0 = 2 + 3t_0$. Therefore,

$$1 = x_0 + y_0 + z_0 = t_0 + (1 - 2t_0) + (2 + 3t_0) = 3 + 2t_0.$$

This implies $t_0 = -1$ and hence $x_0 = -1$, $y_0 = 3$, $z_0 = -1$.

Answer. The intersection is the point $(-1, 3, -1)$.

Problem 3. Find intersection of two planes $-4x + 2y + z = 0$ and $x - z = 5$.

Solution. Normal vectors to planes are determined by coefficients in front of x, y, z, that is, in our case, $\vec{n}_1 = (-4, 2, 1)$, $\vec{n}_2 = (1, 0, -1)$. If a line ℓ lies in both planes then it is orthogonal to both $\vec{n}_1$ and $\vec{n}_2$. Therefore ℓ is parallel to their cross product

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 2 & 1 \\ 1 & 0 & -1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -4 & 1 \\ 1 & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -4 & 2 \\ 1 & 0 \end{vmatrix} = (-2, -3, -2) = -(2, 3, 2).$$

Thus ℓ is parallel to $(2, 3, 2)$. Now we find a point $P_0 = (x_0, y_0, z_0)$ on ℓ. We assume that $z_0 = 0$. As P_0 belongs to both planes (and $z_0 = 0$) we have

$$\begin{cases} -4x_0 + 2y_0 = 0 \\ x_0 = 5, \end{cases}$$

which means that $x_0 = 5$ and $2y_0 = 4x_0 = 20$, so $y_0 = 10$. Therefore, the point $(5, 10, 0)$ belongs to the line ℓ, so we obtain the answer.

Answer. The intersection is the line

$$\frac{x-5}{2} = \frac{y-10}{3} = \frac{z}{2}.$$

Remark. It is worth to **check the answers.**

In Problem 3, let us check that the line from the answer indeed lies in both planes. From $(x-5)/2 = z/2$ we observe that $x = z+5$, so the line lies in the second plane. From $(y-10)/3 = z/2$ we observe that $y = (3/2)z + 10$. Thus

$$-4x + 2y + z = -4z - 20 + 3z + 20 + z = 0,$$

so the line lies in the first plane as well.

In Problem 2, we check that $(-1, 3, -1)$ belongs to the plane $x+y+z = 1$. Indeed, $-1+3-1 = 1$. We also can check that $(-1, 3, -1)$ satisfies equation for the line with $t = -1$.