Quiz # 3.

Problem 1. Find the distance between points $A = (2, 0, -1)$ and $B = (4, 2\sqrt{2}, 1)$. Find the angle between vectors $\vec{AB}$ and $\vec{a} = (4, 0, 3)$. Are the vectors orthogonal?

Solution. $\vec{AB} = (4-2, 2\sqrt{2}-0, 1-(-1)) = (2, 2\sqrt{2}, 2)$ and $|AB| = |\vec{AB}| = \sqrt{2^2 + (2\sqrt{2})^2 + 2^2} = \sqrt{16} = 4$. Denoting the angle between $\vec{AB}$ and $\vec{a}$ by θ, and using formula $\vec{AB} \cdot \vec{a} = |\vec{AB}|\,|\vec{a}| \cos\theta$, we observe that

$$\cos\theta = \frac{\vec{AB} \cdot \vec{a}}{|\vec{AB}|\,|\vec{a}|} = \frac{2 \cdot 4 + 2\sqrt{2} \cdot 0 + 2 \cdot 3}{4\sqrt{4^2 + 0^2 + 3^2}} = \frac{14}{4\sqrt{25}} = \frac{7}{10}.$$

As the dot product is not 0, the vectors are not orthogonal.

Answer. The distance is 4, the angle is $\arccos(7/10)$, the vectors are not orthogonal.

Problem 2. Using the definition only, find the Maclaurin series of $x \sin x$.

Solution. Denote $f(x) = x \sin x$. The $f'(x) = \sin x + x \cos x$, $f''(x) = 2\cos x - x \sin x$, $f'''(x) = -3 \sin x - x \cos x$, $f^{(4)} = -4\cos x + x \sin x$, ... Continuing in this way, we observe that for odd n, say $n = 2k+1$ $(k = 0, 1, 2, 3, 4, ...)$,

$$f^{(n)}(x) = f^{(2k+1)}(x) = (-1)^k \left((2k+1)\sin x + x\cos x\right)$$

and for even n, say $n = 2k$ $(k = 1, 2, 3, 4, ...)$,

$$f^{(n)}(x) = f^{(2k)}(x) = (-1)^{k+1}\left(2k\, \cos x - x \sin x\right).$$

Therefore, for odd n, $f^{(n)}(0) = 0$, while for even $n = 2k$, $f^{(n)}(0) = f^{(2k)}(0) = (-1)^{k+1}\, 2k$ (note also $f(0) = 0$). Thus the Maclaurin series of f is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\, 2k}{(2k)!} x^{2k} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)!} x^{2k} = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \frac{x^{10}}{9!} - \frac{x^{12}}{12!} + ...$$

Answer.
Way 1.

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)!} x^{2k}$$

Way 2.

$$x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \frac{x^{10}}{9!} - \frac{x^{12}}{12!} + ...$$

(both ways are acceptable).

Remarks. 1. Note, to write $x \sin x = \sum_{k=1}^{\infty} ...$ one has to explain why the series is convergent for every x and why the limit is $x \sin x$.

2. Note the solution "since $\sin x = x - x^3/3! + x^5/5! - ...$, we observe $x \sin x = x^2 - x^4/3! + x^6/5! - ...$" is not acceptable, as except definitions it uses also properties of series (the product of a series and a number).