Quiz # 3.

Problem 1. Find the angle between vectors $\vec{a} = (\sqrt{3}, 2\sqrt{3}, -1)$ and $\vec{b} = (1, 2\sqrt{2}, \sqrt{3})$. Are the vectors orthogonal?

Solution. $\vec{a} \cdot \vec{b} = \sqrt{3} + 4\sqrt{6} - \sqrt{3} = 4\sqrt{6} \neq 0$, hence the vectors are not orthogonal. Now, $|\vec{a}| = \sqrt{3 + 4 \cdot 3 + 1} = \sqrt{16} = 4$ and $|\vec{b}| = \sqrt{1 + 4 \cdot 2 + 3} = \sqrt{12}$. Therefore, denoting the angle between $\vec{a}$ and $\vec{b}$ by θ,

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{4\sqrt{6}}{4\sqrt{12}} = \frac{1}{\sqrt{2}} \quad \text{and hence} \quad \theta = \arccos(1/\sqrt{2}) = \pi/4.$$

Answer. The angle is $\pi/4$, the vectors are not orthogonal.

Problem 2. Using the definition, find the Maclaurin series of e^{-3x}.

Solution. Denote $f(x) = e^{-3x}$. The $f'(x) = -3e^{-3x}$, $f''(x) = (-3)^2 e^{-3x}$, $f'''(x) = (-3)^3 e^{-3x}$, ... Continuing in this way we see that $f^{(n)}(x) = (-3)^n e^{-3x}$. Since $e^0 = 1$, $f^{(n)}(0) = (-3)^n$. Therefore the Maclaurin series of f is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, x^n = \sum_{n=0}^{\infty} \frac{(-3)^n}{n!}\, x^n.$$

Answer. $\sum_{n=0}^{\infty} \frac{(-3)^n}{n!}\, x^n$.

Remark. Note, to write $f(x) = \sum_{n=0}^{\infty} \frac{(-3)^n}{n!}\, x^n$, one should explain why the series is convergent for every x and why the limit is $f(x)$. This can be done using Taylor's Theorem (note that $|f^{(n)}(0)| \leq 3^n$, so $|f^{(n)}(0)x^n| \leq (3R)^n$ on any fixed interval $(-R, R)$).

Problem 3. Evaluate the indefinite integral $\int x^4 e^{-3x} dx$ as an infinite series (don't use the integration by parts). You may assume that the Maclaurin series converges to the function.

Solution. Note that in the previous problem, we observed that the Maclaurin series of $x^4 e^{-3x}$ is

$$x^4 \sum_{n=0}^{\infty} \frac{(-3)^n}{n!}\, x^n = \sum_{n=0}^{\infty} \frac{(-3)^n}{n!}\, x^{n+4}.$$

Since the Maclaurin series converges to the function,

$$\int x^4 e^{-3x} dx = \int \sum_{n=0}^{\infty} \frac{(-3)^n}{n!}\, x^{n+4}\, dx.$$

Using integration of power series and that $\int x^{n+4}\, dx = C + \frac{x^{n+5}}{n+5}$, we obtain the following answer.

Answer.

$$\int x^4 e^{-3x} dx = C + \sum_{n=0}^{\infty} \frac{(-3)^n}{(n+5)\, n!}\, x^{n+5} = C + \sum_{n=5}^{\infty} \frac{(-3)^{n-5}}{n\,(n-5)!}\, x^n.$$