Quiz # 2.

Problem 1. Test the following series for convergence or divergence

$$\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n^3}\right).$$

Solution. a. We compare the series with $\sum_{n=1}^{\infty} 1/n^2$. Denote $x = 1/n^3 \to 0$ as $n \to \infty$ and note that

$$\lim_{n\to\infty} \frac{n \sin(1/n^3)}{1/n^2} = \lim_{n\to\infty} \frac{\sin(1/n^3)}{1/n^3} = \lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Since $\sum_{n=1}^{\infty} 1/n^2$ is convergent, by the limit comparison test, $\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n^3}\right)$ is convergent as well.

Answer. The series is convergent.

Problem 2. Find the radius of convergence and the interval of convergence of the following series

$$\sum_{n=1}^{\infty} \frac{(x+1)^n}{\sqrt{n}\, 3^n}.$$

Solution. To find the radius of convergence we apply the Ratio Test. Denote $a_n = \frac{(x+1)^n}{\sqrt{n}\,3^n}$. Then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x+1)^{n+1} \sqrt{n}\, 3^n}{\sqrt{n+1}\, 3^{n+1} (x+1)^n}\right| = \lim_{n\to\infty} \left|\frac{(x+1)\sqrt{n}}{3\sqrt{n+1}}\right| = \frac{|x+1|}{3} \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = \frac{|x+1|}{3}.$$

By the Ratio Test the series (of absolute values) is convergent for $|x+1|/3 < 1$ (that is $|x+1| < 3$, which means $-4 < x < 2$) and divergent for $|x+1|/3 > 1$. Thus the radius of convergence is 3 and the series is convergent at least for $x \in (-4, 2)$. Now we check the end points. For $x = 2$ we have

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{3^n}{\sqrt{n}\, 3^n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.$$

This is the p-series with $p = 1/2 < 1$, which diverges (or apply the integral test). For $x = -4$ we have

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-3)^n}{\sqrt{n}\, 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}.$$

This an alternating series, which converges as $1/\sqrt{n}$ is decreasing and goes to 0 as $n \to \infty$.

Thus the interval of convergence is $[-4, 2)$.

Answer. The radius of convergence is 3, the interval of convergence is $[-4, 2)$.