Quiz # 2.

Problem 1. Test the series for convergence or divergence

$$\textbf{a.} \quad \sum_{n=1}^{\infty} \cos(\pi/n) \qquad\qquad \textbf{b.} \quad \sum_{n=1}^{\infty} \sin(\pi/n)$$

Solution. a. Note that $\pi/n \to 0$ as $n \to \infty$. Since cos is a continuous function, we also observe that $\cos(\pi/n) \to \cos 0 = 1 \neq 0$ as $n \to \infty$. It shows that the series is divergent by the Test for Divergence.
b. We compare our series with the divergent series $\sum_{n=1}^{\infty} \frac{1}{n}$ (it was shown in the class that this series diverges; it can be also checked by the integral test).

It is known (and can be checked by l'Hospital Rule) that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Since the function $(\sin x)/x$ is continuous and $\pi/n \to 0$ as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \frac{\sin(\pi/n)}{1/n} = \pi \lim_{n\to\infty} \frac{\sin(\pi/n)}{\pi/n} = \pi.$$

Since the limit is positive and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, we conclude that our series is divergent by the Limit Comparison Test.

Answer. Both series are divergent.

Problem 2. Find the radius of convergence and the interval of convergence of the following series

$$\sum_{n=1}^{\infty} \frac{(-1)^n\,(x-2)^n}{n}$$

Solution. To find the radius of convergence we apply the Ratio Test. Denote $a_n = \frac{(-1)^n\,(x-2)^n}{n}$. Then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(x-2)^{n+1}\,n}{(n+1)\,(x-2)^n}\right| = \lim_{n\to\infty} \left|\frac{(x-2)\,n}{(n+1)}\right| = |x-2| \lim_{n\to\infty} \left|\frac{n}{(n+1)}\right| = |x-2|.$$

By the Ratio Test the series (of absolute values) is convergent for $|x-2| < 1$ and divergent for $|x-2| > 1$. Thus the radius of convergence is 1 and the series is convergent for $x \in (1,3)$. Now we check the end points. For $x = 3$ we have

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which is an alternating series. Since $1/n \to 0$ as $n \to \infty$ this series is convergent. Finally, if $x = 1$ then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n\,(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n},$$

which diverges (as we noticed in Problem 1). This implies that the interval of convergence is $(1,3]$.

Answer. The radius of convergence is 1, the interval of convergence is $(1,3]$.