University of Alberta
Math214

Functions of multiple variables

Exercise 1. Check that the origin is a critical point for each of the following functions, and use the second derivative test to see if it is a local maximum, local minimum, or a saddle point.

1. $f(x,y) = x^2 + xy + y^2 + x^3 + y^3$.
2. $f(x,y) = x^2 + 3xy + y^2 + \sin^3(x)$.
3. $f(x,y) = \sin(xy)$.
4. $f(x,y) = \sin(x^2 + y^2)$.

Exercise 2. Find all critical points of the function

$$f(x,y) = x^2 + xy + y^2 - 4x - 5y + 5$$

and check each to see if it's a max, min, or saddle point.

Solutions

Exercise 1.

1. First let us calculate the partial derivatives.
   we have $f_x(x,y) = 2x + y + 3x^2$ and $f_y(x,y) = x + 2y + 3y^2$. Since $f_x(0,0) = f_y(0,0) = 0$ then $(0,0)$ is a critical point. In order to apply the second derivative test, let us calculate the second order derivatives.
   We have $f_{xx}(x,y) = 2 + 6x$, $f_{yy}(x,y) = 2 + 6y$ and $f_{xy}(x,y) = 1$. Now $D(x,y) = f_{xx}(x,y) \cdot f_{yy}(x,y) - (f_{xy}(x,y))^2$. At $(0,0)$, we get $D(0,0) = 3 > 0$. Since also $f_{xx}(0,0) > 0$ then f has a local minimum at $(0,0)$.

2. First let us calculate the partial derivatives.
   we have $f_x(x,y) = 2x + 3y + 3\sin^2(x)\cos(x)$ and $f_y(x,y) = 3x + 2y$. Since $f_x(0,0) = f_y(0,0) = 0$ then $(0,0)$ is a critical point. In order to apply the second derivative test, let us calculate the second order derivatives.
   We have $f_{xx}(x,y) = 2 + 6\sin(x)\cos^2(x) - 3\sin^3(x)$, $f_{yy}(x,y) = 2$ and $f_{xy}(x,y) = 3$. Now $D(x,y) = f_{xx}(x,y) \cdot f_{yy}(x,y) - (f_{xy}(x,y))^2$. At $(0,0)$, we get $D(0,0) = -5 < 0$. This means that f does not have a local minimum at $(0,0)$ and this point is a saddle point.

3. First let us calculate the partial derivatives.
   we have $f_x(x,y) = y\cos(xy)$ and $f_y(x,y) = x\cos(xy)$. Since $f_x(0,0) = f_y(0,0) = 0$ then $(0,0)$ is a critical point. In order to apply the second derivative test, let us calculate the second order derivatives.
   We have $f_{xx}(x,y) = -y^2\sin(xy)$, $f_{yy}(x,y) = -x^2\sin(xy)$ and $f_{xy}(x,y) = \cos(xy) - xy\sin(xy)$. Now $D(x,y) = f_{xx}(x,y) \cdot f_{yy}(x,y) - (f_{xy}(x,y))^2$. At $(0,0)$, we get $D(0,0) = -1 < 0$. This means that f does not have a local minimum at $(0,0)$ and this point is a saddle point.

4. First let us calculate the partial derivatives.
   we have $f_x(x,y) = 2x\cos(x^2+y^2)$ and $f_y(x,y) = 2y\cos(x^2+y^2)$. Since $f_x(0,0) = f_y(0,0) = 0$ then $(0,0)$ is a critical point. In order to apply the second derivative test, let us calculate the second order derivatives.
   We have $f_{xx}(x,y) = 2\cos(x^2+y^2) - 4x^2\sin(x^2+y^2)$, $f_{yy}(x,y) = 2\cos(x^2+y^2) - 4y^2\sin(x^2+y^2)$ and $f_{xy}(x,y) = -4xy\sin(x^2+y^2)$. Now $D(x,y) = f_{xx}(x,y) \cdot f_{yy}(x,y) - (f_{xy}(x,y))^2$. At $(0,0)$, we get $D(0,0) = 4 > 0$. Since also $f_{xx}(0,0) > 0$ then f has a local minimum at $(0,0)$.

Exercise 2. First let us calculate the partial derivatives.
we have $f_x(x,y) = 2x + y - 4$ and $f_y(x,y) = 2y + x - 5$. To find the critical points we need to solve the system given by $f_x(x,y) = f_y(x,y) = 0$. Solving the system we find $x = 1$ and $y = 2$. This means that $(1,2)$ is a critical point. In order to apply the second derivative test, let us calculate the second order derivatives.
We have $f_{xx}(x,y) = 2$, $f_{yy}(x,y) = 2$ and $f_{xy}(x,y) = 1$. Now $D(x,y) = f_{xx}(x,y) \cdot f_{yy}(x,y) - (f_{xy}(x,y))^2$. At $(1,2)$, we get $D(0,0) = 3 > 0$. Since also $f_{xx}(0,0) > 0$ then f has a local minimum at $(1,2)$.