MATH 214 A1, Fall 2013, Solutions to Practice Questions 1

1. (a) The sequence diverges to ∞ because

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2 - 1/n}{n+1} \lim_{n\to\infty} \frac{n^2(1-1/n^3)}{n(1+1/n)} = \lim_{n\to\infty} n \frac{1-1/n^3}{1+1/n} = \infty.$$

   (b) The sequence converges with $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n+\sin(n)/n} = 0$ for the following reason: Since $\sin(n)$ is bounded, we have $\lim_{n\to\infty} \sin(n)/n = 0$ $\big($due to the Squeeze Theorem with $-\frac{1}{n} \le \frac{\sin(n)}{n} \le \frac{1}{n}\big)$ and hence $\lim_{n\to\infty}(n + \sin(n)/n) = \infty$.

   (c) Using

$$-\frac{1}{n^2} \le \frac{(-1)^n}{n^2} \le \frac{1}{n^2}$$

   and $\lim_{n\to\infty}\left(-\frac{1}{n^2}\right) = \lim_{n\to\infty} \frac{1}{n^2} = 0$, it follows from the Squeeze Theorem that

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{(-1)^n}{n^2} = 0.$$

   (d) This is of the form "∞^0", which is indeterminate, so we must be careful. We have

$$\lim_{n\to\infty} n^{-1/n} = \lim_{x\to\infty} x^{-1/x} = \exp\left(-\lim_{x\to\infty} \frac{\ln(x)}{x}\right)$$

   provided that the limit exists. L'Hospital's rule yields

$$\lim_{x\to\infty} \frac{\ln(x)}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

   Therefore, the sequence converges with $\lim_{n\to\infty} d_n = \mathrm{e}^0 = 1$.

2. (a) We have $a_n = r^n$ for $r = -\frac{3}{4}$. Since $|r| < 1$, we obtain

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(-\frac{3}{4}\right)^n = 0.$$

   (b) Since $\frac{4}{3} > 1$, the sequence diverges.

   (c) We have $c_n = \cos(\pi n) = (-1)^n$, which is a divergent sequence.

   (d) We have $d_n = \sin(\pi n) = 0$, hence $\lim_{n\to\infty} d_n = 0$.

3. (a) This is a geometric series with

$$\sum_{n=2}^{\infty}(-3)^{n-1}4^{-n} = \sum_{j=1}^{\infty}(-3)^{j}4^{-(j+1)} = \sum_{j=1}^{\infty}\frac{-3}{16}\left(-\frac{3}{4}\right)^{j-1},$$

hence $\sum_{n=1}^{\infty} ar^{n-1}$ with $a = -3/16$ and $r = -3/4$. Since $|r| < 1$, the series converges and $\sum_{n=1}^{\infty} ar^{n-1} = a\frac{1}{1-r} = \frac{-3}{16}\frac{1}{7/4} = -\frac{3}{28}$.

(b) This is a geometric series with

$$\sum_{n=0}^{\infty}(-4)^{n-1}3^{-n} = \sum_{j=1}^{\infty}(-4)^{j-2}3^{-(j-1)} = \sum_{j=1}^{\infty}\frac{-1}{4}\left(-\frac{4}{3}\right)^{j-1},$$

hence $\sum_{n=1}^{\infty} ar^{n-1}$ with $a = -1/4$ and $r = -4/3$. Since $|r| > 1$, the series diverges.

(c) Because $\lim_{n\to\infty} 3^{1/n}4^{n} = \infty$, the series diverges.

(d) Because $\lim_{n\to\infty} 3^{1/n}4^{1/n} = \lim_{n\to\infty} 12^{1/n} = 12^{\lim_{n\to\infty} 1/n} = 12^{0} = 1 \neq 0$, the series diverges.